Invariance Times and BSDEs Stopped Before a Random Time

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Based on

Crépey, S. and S. Song (2017).
Invariance times.
Annals of Probability.

Crépey, S. and S. Song (2015).
BSDEs of counterparty risk.
Stochastic Processes and their Applications.

Albanese, C. and S. Crépey (2017).
XVA analysis from the balance sheet
Working paper on my webpage

Crépey, S. and S. Song (2016).
Counterparty risk and funding: Immersion and beyond.
Finance and Stochastics (long preprint version on my webpage).

Works in progress..

Notation

- For any process Y and times T and θ
  - $Y^T = Y\mathbb{1}_{[0,T)} + Y_T\mathbb{1}_{[T,+\infty)}$
  - Assuming Y left-limited:
    $Y^{\theta-} = Y\mathbb{1}_{[0,\theta)} + Y_{\theta-}\mathbb{1}_{[\theta,+\infty)}$
    $Y^{\theta-\wedge T} = (Y^{\theta-})^T = (Y^T)^{\theta-}$,
- $\mathcal{S}_{\mathcal{I}}(\mathbb{G},\mathbb{Q}), \mathcal{M}_{\mathcal{I}}(\mathbb{G},\mathbb{Q})$ Semimartingales, local martingales on a predictable interval $\mathcal{I}$
  - $\mathcal{I} = \mathbb{R}_+$ by default

Mathematical viewpoint

Finding conditions under which the progressive enlargement of filtrations Jeulin-Yor formula can be compensated by the Girsanov formula of an equivalent change of probability measure

Counterparty risk motivation

- Counterparty risk is related to cash flows or valuations linked to either counterparty default or the default of the bank itself.
- C. Albanese and S. C. XVA analysis from the balance sheet: A key distinction is between
  - the cash flows received by the bank prior its default time θ
  - the cash flows received by the bank during the default resolution period starting at θ.
- The first stream of cash flows affects the bank shareholders, whereas the second stream of cash flows only affects creditors.

- For accepting a new deal, shareholders need to be at least indifferent given the cash flows before θ only.

$\rightarrow$ In the context of so-called XVA analysis, all the cumulative cash flow and gain processes need to be stopped before the default time θ of the bank itself.

$\rightarrow$ XVA BSDEs stopped before a terminal time

Full BSDE

Given

- a fixed time horizon (final maturity) $T > 0$
- a totally inaccessible stopping time θ,
- a running cost $g_t(\omega, y)$,

we consider the following BSDE in $Y \in \mathcal{S}(\mathbb{G}, \mathbb{Q})$:

$$\begin{cases} Y_T \mathbb{1}_{\{T<\theta\}} = 0, \\ Y_t^{\theta- \wedge T} + \int_0^{t\wedge\theta\wedge T} g_s(Y_{s-})ds \in \mathcal{M}(\mathbb{G}, \mathbb{Q}). \end{cases} \tag{1}$$

Extensions to BSDEs

- with $g = g_t(y, z)$, where additional arguments z correspond to integrands in a stochastic integral representation of the martingale part of Y
- with further finite variation (non-necessarily Lebesgue absolutely continuous) driving terms
- ...

"Duffie, Schroder, and Skiadas (1996)'s" solution

- Forget θ in (1) ("send it to infinity"), obtain a solution $\bar{Y}$ of the resulting (simpler) equation and then set $Y = \bar{Y}^{\theta-}$
- Only yields a solution Y to (1) if $\bar{Y}$ does not jump at θ

Reduction of filtration

Let θ be a $\mathbb{G}$ stopping time (not necessarily totally inaccessible), and let $\mathbb{F}$ be a subfiltration of $\mathbb{G}$ ($\mathbb{F}$ and $\mathbb{G}$ both satisfying the usual conditions), such that:

Condition (B)

For any $\mathbb{G}$ predictable process L, there exists an $\mathbb{F}$ predictable process L', which we call the $\mathbb{F}$ predictable reduction of L, such that L' coincides with L until θ.

Lemma 1 (“If” part = Lemma 1 in Jeulin and Yor (1978))

The filtration $\mathbb{F}$ *satisfies the condition (B) if and only if* $\mathbb{G}$ *is a subfiltration of* $\overline{\mathbb{F}} = (\overline{\mathcal{F}}_t)_{t \in \mathbb{R}_+}$, *where*

$$\overline{\mathcal{F}}_t = \{B \in \mathcal{A} : \exists A \in \mathcal{F}_t, A \cap \{t < \theta\} = B \cap \{t < \theta\}\}.$$

- The condition (B) relaxes the standard progressive enlargement of filtration setup where $\mathbb{G}$ reduces to the smallest filtration containing $\mathbb{F}$ for which θ is a stopping time
- All the classical results of progressive enlargement of filtration are still valid under (B):
  - "Key lemma of credit risk",
  - Existence of $\mathbb{F}$ optional reductions (also denoted with $'$) coinciding with $\mathbb{G}$ optional processes before θ,
  - ...

- In particular, let S represent the $\mathbb{F}$ Azéma supermartingale of θ, i.e. $S_t = \mathbb{Q}(\theta > t \,|\mathcal{F}_t)$, $t > 0$, with Doob-Meyer decomposition $S = Q - D$.
- The Jeulin-Yor theorem says that for any bounded $(\mathbb{F}, \mathbb{Q})$ martingale X, the process

$$X^{\theta-} - \frac{\mathbb{1}_{(0,\theta]}}{S_-} \cdot \langle S, X\rangle^{(\mathbb{F},\mathbb{Q})}$$

is a $(\mathbb{G}, \mathbb{Q})$ uniformly integrable martingale.

The direct part in the next result addresses the "inverse problem" of knowing when an $(\mathbb{F}, \mathbb{Q})$ semimartingale X is such that $X^{\theta-}$ is a $(\mathbb{G}, \mathbb{Q})$ local martingale.

Lemma 2 (Song (2014))

- *For any* $X \in \mathcal{S}_{\{\mathsf{S}_- > 0\}}(\mathbb{F}, \mathbb{Q})$ *such that* $\mathsf{S}_- \cdot X + [\mathsf{S}, X] \in \mathcal{M}_{\{\mathsf{S}_- > 0\}}(\mathbb{F}, \mathbb{Q})$, *then* $X^{\theta-} \in \mathcal{M}(\mathbb{G}, \mathbb{Q})$.
- *Conversely, for any* $M \in \mathcal{M}(\mathbb{G}, \mathbb{Q})$ *with* $\Delta_\theta M = 0$, *then* $M' \in \mathcal{S}_{\{\mathsf{S}_- > 0\}}(\mathbb{F}, \mathbb{Q})$ *and* $\mathsf{S}_- \cdot M' + [\mathsf{S}, M'] \in \mathcal{M}_{\{\mathsf{S}_- > 0\}}(\mathbb{F}, \mathbb{Q})$.

- "Immersion case" (of a pseudo-stopping time avoiding $\mathbb{F}$ stopping times) where S is continuous and nonincreasing: Then $[\mathsf{S}, \cdot] = 0$, so that the martingale conditions in the above reduce to X, resp. M' $\in \mathcal{M}_{\{\mathsf{S}_- > 0\}}(\mathbb{F}, \mathbb{Q})$.

- The Jeulin formula and Lemma 2 can be viewed as progressive enlargement formal analogs of the predictable and optional Girsanov measure change formulas, the Azéma supermartingale S playing the role of the measure change density from the probability measure $\mathbb{Q}$ to some $\mathbb{P} \ll \mathbb{Q}$
  - Predictable Girsanov formula
    "For any bounded $X \in \mathcal{M}(\mathbb{F}, \mathbb{Q})$, $X - \frac{1}{\mathsf{S}_-} \centerdot \langle S, X \rangle^{(\mathbb{F},\mathbb{Q})} \in \mathcal{M}(\mathbb{F}, \mathbb{P})$"
  - Optional Girsanov formula
    "$X \in \mathcal{M}(\mathbb{F}, \mathbb{P})$ iff $X \in \mathcal{S}_{\{\mathsf{S}_- > 0\}}(\mathbb{F}, \mathbb{Q})$ and $\mathsf{S}_- \centerdot X + [\mathsf{S}, X] \in \mathcal{M}_{\{\mathsf{S}_- > 0\}}(\mathbb{F}, \mathbb{Q})$"

Recall $S = \mathrm{Q} - \mathrm{D}$ denotes the $(\mathbb{F}, \mathbb{Q})$ canonical Doob-Meyer decomposition of $\mathrm{S}_t = \mathbb{Q}(\theta > t \,|\mathcal{F}_t),\; t > 0$.

Lemma 3 (Song (2014))

One has the following unique predictable multiplicative decomposition of the Azéma supermartingale S *of* θ *on* $\{{}^p\mathrm{S} > 0\}$*:*

$$\mathrm{S} = \mathrm{S}_0 \mathcal{E}(-\frac{1}{\mathrm{S}_-} \centerdot \mathrm{D}) \mathcal{E}(\frac{1}{{}^p\mathrm{S}} \centerdot \mathrm{Q}).$$

Reduced BSDE

- Recall the full BSDE with running cost g and terminal time θ (if $< T$)
- Letting $U = Y'$, the "$(\mathbb{G}, \mathbb{Q})$ local martingale-to be" in the full BSDE satisfies

$$
\begin{aligned}
Y_t^{\theta - \wedge T} + \int_0^{t \wedge \theta \wedge T} g_s(Y_{s-}) ds &= U_t^{\theta - \wedge T} + \int_0^{t \wedge \theta \wedge T} g'_s(U_{s-}) ds \\
&= \Big(\underbrace{U^T + \int_0^{\cdot \wedge T} g'_s(U^T_{s-}) ds}_{\overline{U}} \Big)_t^{\theta -}.
\end{aligned}
$$

This suggests to solve the full BSDE with Lemma 2.

Namely, we consider the following BSDE for $U \in \mathcal{S}_{\{\mathsf{S}_- > 0\}}(\mathbb{F}, \mathbb{Q})$:

$$U_T \mathsf{S}_T = 0\,,\;\; \mathsf{S}_- \cdot \overline{U} + [\mathsf{S}, \overline{U}] \in \mathcal{M}_{\{\mathsf{S}_- > 0\}}(\mathbb{F}, \mathbb{Q}). \tag{2}$$

Proposition 1

The BSDEs (1) *and* (2) *are equivalent. Specifically:*
- *If Y is a solution to the BSDE* (1)*, then $U = Y'$ is a solution to the BSDE* (2)*.*
- *Conversely, if U is a solution to the BSDE* (2)*, then $Y = U^{\theta-}$ is a solution to the BSDE* (1)*.*

"Immersion case", where S is continuous and nonincreasing, of a pseudo-stopping time avoiding $\mathbb{F}$ stopping times

- Then $[\mathsf{S}, \cdot] = 0$, so that the martingale condition in (2) reduces to $\overline{U} \in \mathcal{M}_{\{\mathsf{S}_- > 0\}}(\mathbb{F}, \mathbb{Q})$.

Condition (A)

- However, beyond this "immersion" case, even if passing from (1) to (2) allows removing θ from the equation, this comes at the expense of a more involved martingale condition in (2)
- As a shortcut out of this, suppose

Condition (A)

There exists a probability measure $\mathbb{P}$ equivalent to $\mathbb{Q}$ on $\mathcal{F}_T$ such that, for any $(\mathbb{F},\mathbb{P})$ local martingale P, $P^{\theta-}$ is a $(\mathbb{G},\mathbb{Q})$ local martingale on $[0,T]$.

- Then any solution $U \in \mathcal{S}_{\{\mathsf{S}_->0\}}(\mathbb{F},\mathbb{P})$ to the "reduced BSDE"

$$U_T \mathsf{S}_T = 0\,,\ \ \overline{U}_t = U_t^T + \int_0^{t\wedge T} g'_s(U_{s-})ds \in \mathcal{M}_{\{\mathsf{S}_->0\}}(\mathbb{F},\mathbb{P}) \qquad (3)$$

  yields a solution $Y = U^{\theta-}$ to (1).
  - Noting $\mathsf{S}_{\theta-} > 0$ on $\{0 < \theta < \infty\}$

Invariance times

- This allows getting rid of θ in the BSDE
- Modulo reduction, the martingale condition in the reduced BSDE (3) is exactly the same as the one in the full BSDE (1).
- As the running costs are integrated until the end of times T in the reduced BSDE, this results in pricing rules consistent with the prescription of the regulator, which says quite explicitly that a bank capital cannot be seen increasing as a consequence of the sole deterioration of the bank credit, all else being equal
  - UCVA versus first-to-default CVA

Definition 1

If the condition (A) is satisfied, we call the random time θ an invariance time and the related probability measure $\mathbb{P}$ an invariance probability measure.

- Stopping before θ rather than at θ in the condition (A) appears naturally in our counterparty risk application and BSDE motivation.
- However, the condition (A), with the "stopping before θ" operator, is nonstandard in the enlargement of filtration literature.

$\rightarrow$ Strength of the condition (A)?

$\rightarrow$ Equivalence, under the condition (A), between the full BSDE (1) and the reduced BSDE (3)??

Main result

Given a probability measure $\mathbb{P}$ equivalent to $\mathbb{Q}$ on $\mathcal{F}_T$, let $q = \frac{1}{p}$ denote the $(\mathbb{F}, \mathbb{Q})$ martingale density function $\left.\frac{d\mathbb{P}}{d\mathbb{Q}}\right|_{\mathcal{F}_{t \wedge T}}$, $t \in \mathbb{R}_+$.

Theorem 1

Assuming the condition (B) on $\mathbb{F}$ and given a constant $T > 0$:

(i) *A probability measure $\mathbb{P}$ equivalent to $\mathbb{Q}$ on $\mathcal{F}_T$ is an invariance probability measure if and only if*

$$q = q_0 \mathcal{E}(\tfrac{1}{{}^p\mathsf{S}} \centerdot \mathsf{Q}) \text{ on } \{{}^p\mathsf{S} > 0\} \cap [0, T] \tag{4}$$

(ii) *The condition (A) holds if and only if*

$$\mathcal{E}(\mathbb{1}_{\{{}^p\mathsf{S}>0\}} \tfrac{1}{{}^p\mathsf{S}} \centerdot \mathsf{Q}) \text{ is a positive } (\mathbb{F}, \mathbb{Q}) \text{ true martingale on } [0, T]. \tag{5}$$

In this case, an invariance probability measure $\mathbb{P}$ is given by the $\mathbb{Q}$ density

$$\mathcal{E}(\mathbb{1}_{\{{}^p\mathsf{S}>0\}} \frac{1}{{}^p\mathsf{S}} \centerdot \mathsf{Q})_T \tag{6}$$

Proof idea. The measure change "compensates" the reduction of filtration.
Start with, for any $P \in \mathcal{M}(\mathbb{F}, \mathbb{P})$

$$\underbrace{\underbrace{(P - q \centerdot [p, P])}_{\text{Girsanov}}{}^{\theta-} - \frac{\mathbb{1}_{(0,\theta]}}{\mathsf{S}_-} \centerdot \langle \mathsf{Q}, P - q \centerdot [p, P] \rangle}_{\text{Jeulin}-\text{Yor}} \in \mathcal{M}_{[0,T]}(\mathbb{G}, \mathbb{Q}).$$

But

$$\begin{aligned} &(P - q \centerdot [p, P])^{\theta-} - \frac{\mathbb{1}_{(0,\theta]}}{\mathsf{S}_-} \centerdot \langle \mathsf{Q}, P - q \centerdot [p, P] \rangle \\ &\qquad = P^{\theta-} - \left(\mathbb{1}_{[0,\theta)} q \centerdot [p, P] + \frac{\mathbb{1}_{(0,\theta]}}{\mathsf{S}_-} \centerdot \langle \mathsf{Q}, P - q \centerdot [p, P] \rangle \right). \end{aligned}$$

Hence $\mathbb{P}$ is an invariance probability measure iff

$$\left(\mathbb{1}_{[0,\theta)} q \centerdot [p, P] + \frac{\mathbb{1}_{(0,\theta]}}{\mathsf{S}_-} \centerdot \langle \mathsf{Q}, P - q \centerdot [p, P] \rangle \right) \in \mathcal{M}_{[0,T]}(\mathbb{G}, \mathbb{Q}), \quad \forall P \in \mathcal{M}(\mathbb{F}, \mathbb{P})$$

Then project, use density arguments, etc. ∎

Proposition 2

- *If $\mathbb{F} = \mathbb{G}$ and θ has an intensity, then θ cannot be an invariance time unless $\mathbb{Q}(\theta \leq T) = 0$.*
- *Given $\mathbb{F} \subseteq \mathbb{G}$ satisfying (B), $\mathbb{P} = \mathbb{Q}$ is an invariance probability measure for all $T > 0$ if and only if $\mathsf{Q} = \mathsf{S}_0$.*

Proof. 1) In the case where $\mathbb{F} = \mathbb{G}$ and θ has an intensity, we have

$$\mathsf{S} = \mathbb{1}_{[0,\theta)}\,, \;\; \mathsf{D} \text{ is continuous }\,, \;\; {}^p\mathsf{S} = \mathbb{1}_{[0,\theta]},\;\; \mathsf{Q} = \mathbb{1}_{[0,\theta)} + \mathsf{D} \text{ and } \mathsf{Q}_0 = \mathsf{S}_0 = 1.$$

Hence, using the stochastic exponential formula

$$\begin{aligned}\mathcal{E}(\mathbb{1}_{\{{}^p\mathsf{S}>0\}}\frac{1}{{}^p\mathsf{S}} \centerdot \mathsf{Q})_t =& \mathcal{E}(\mathsf{Q})_t = e^{\mathsf{Q}_t - \mathsf{Q}_0}\prod_{s\leq t}(1+\Delta_s\mathsf{Q})e^{-\Delta_s\mathsf{Q}} \\ &= e^{\mathbb{1}_{\{t<\theta\}}+\mathsf{D}_t - 1}\mathbb{1}_{\{t<\theta\}} = e^{\mathsf{Q}_t}\mathbb{1}_{\{t<\theta\}},\end{aligned}$$

which vanishes at θ on $\{\theta \leq T\}$. Therefore, in view of Theorem 1, the condition (A) cannot hold on $[0, T]$ unless $\mathbb{Q}(\theta \leq T) = 0$.
2) In the case where $\mathbb{P} = \mathbb{Q}$, we have $q = q_0$ on $[0, T]$. Hence, in view of Theorem 1, $\mathbb{P}$ is an invariance probability measure for all $T > 0$ if and only if Q is constant on $[0, T]$. ∎

Example 1

Let $\mathbb{G}$ be the augmentation of the natural filtration of the jump process at an exponential time θ relative to some probability measure $\mathbb{Q}$.

- For $\mathbb{F} = \mathbb{G}$
  - The condition (B) holds trivially
  - Proposition 2 1) shows that the condition (A) does not hold
- For $\mathbb{F}$ trivial
  - As any $\mathbb{G}$ predictable process coincides with a Borel function before θ, the condition (B) is satisfied.
  - The only $(\mathbb{F} = \{\emptyset, \Omega\}, \mathbb{Q})$ local martingales are the constants, so that $\mathbb{P} = \mathbb{Q}$ is an invariance probability measure and θ is an invariance time.

- Theorem 1 shows that the condition (A) reduces to a mild integrability condition.
- In addition, Theorem 1 can be used to establish that, under the condition (A) (and assuming that θ has an intensity),
  the full and reduced BSDEs are equivalent.

Intensity based credit risk pricing formulas

Invariance times also allow extending the progressive enlargement credit risk pricing formulas beyond the restrictive "immersion" (or pseudo-stopping times) setup:

Theorem 2

Under the condition (A), if θ has an intensity γ and $S_T > 0$:

- *For any nonnegative $\mathcal{F}_t$ measurable random variable χ:*

$$\mathbb{E}[\chi \mathbb{1}_{\{T<\theta\}}] = \mathbb{E}^{\mathbb{P}}[\chi e^{-\int_0^T \gamma'_u du}]. \tag{7}$$

- *For any nonnegative $\mathbb{F}$ predictable process K,*

$$\mathbb{E}[K_\theta \mathbb{1}_{\{\theta \le T\}}] = \mathbb{E}^{\mathbb{P}}[\int_0^T K_s e^{-\int_0^s \gamma'_u du} \gamma'_s \, ds].$$

→ Intensity models of credit risk with strong (adverse) dependence between credit risk and the underlying market exposure

- Wrong way and gap risk modeling

Connection with the survival measure

- Assuming that θ has a $(\mathbb{G}, \mathbb{Q})$ intensity γ such that $e^{\int_0^\theta \gamma_t dt}$ is $\mathbb{Q}$ integrable, Collin-Dufresne, Goldstein, and Hugonnier (2004) introduce the "survival measure" $\mathbb{S}$ with the $(\mathbb{G}, \mathbb{Q})$ density process $e^{\int_0^\cdot \gamma_t dt} \mathbb{1}_{[0,\theta)}$.
- Under this measure $\mathbb{S}$, Collin-Dufresne et al. (2004) are also able to derive a rather general intensity based credit risk pricing formula, exempt from the Duffie et al. (1996)'s no-jump condition.

Theorem 3

Under the condition (B), if $S_T > 0$ *and* θ *has a* $(\mathbb{G}, \mathbb{Q})$ *intensity* γ *such that* $e^{\int_0^{\theta} \gamma_t dt}$ *is* $\mathbb{Q}$ *integrable, then* θ *is an invariance time and the restriction to* $\mathcal{F}_T$ *of the invariance probability measures* $\mathbb{P}$ *and of the survival measure* $\mathbb{S}$ *coincide.*

- Collin-Dufresne et al. (2004)'s solution touches the filtration as little as possible but singularly changes the measure
- Invariance times do the opposite and yield a full semimartingale calculus, not only transfer of expectation formulas

Connection with pseudo-stopping times

- This part gives examples which illustrate how the condition (A) can be satisfied in cases where $(\mathbb{F}, \mathbb{P})$ martingales really jump at θ, as well as the connection between the condition (A) and the notion of pseudo-stopping time in Nikeghbali and Yor (2005).
- Consider a $(0, +\infty)$ valued random time θ. It is an $(\mathbb{F}, \mathbb{Q})$ pseudo-stopping time if and only if X^{θ} is a $(\mathbb{G}, \mathbb{Q})$ uniformly integrable martingale for any bounded $\mathbb{F}$ martingale X (cf. Nikeghbali and Yor (2005)).
- Clearly, if a pseudo-stopping time θ avoids the $\mathbb{F}$ stopping times, then it is an invariance time satisfying the condition (A) for any positive constant T, with invariance probability measure $\mathbb{P} = \mathbb{Q}$.

- Let A denote the $\mathbb{F}$ dual optional projection of $\mathbb{1}_{[\theta,\infty)}$. Nikeghbali and Yor (2005) show that θ is a pseudo-stopping time if and only if $\mathsf{S} = 1 - \mathsf{A}$.
- Instead, Proposition 2 2) shows that $\mathbb{P} = \mathbb{Q}$ is an invariance probability measure for any positive constant T if and only if $\mathsf{S} = 1 - \mathsf{D}$
    - Noting that $S_0 = 1$ here, as $\theta > 0$.
- Both conditions coincide if and only if $\mathsf{A} = \mathsf{D}$.
- In the case where θ is a $\mathbb{G}$ totally inaccessible stopping time, $\mathsf{A} = \mathsf{D}$ if and only if θ avoids the $\mathbb{F}$ stopping times.
- Hence, for a $(0, +\infty)$ valued $\mathbb{G}$ totally inaccessible stopping time θ, there are two "orthogonal" cases:
    - If θ has the avoidance property, then θ is a pseudo-stopping time if and only if $\mathbb{Q}$ is an invariance probability measure;
    - If θ does not have the avoidance property, then θ cannot be a pseudo-stopping time and $\mathbb{Q}$ be an invariance probability measure simultaneously.

- The difference is due to the fact that a pseudo-stopping time is defined in terms of stopping at θ, whereas invariance is defined in terms of stopping before θ.
- Having said this regarding the case where $\mathbb{P} = \mathbb{Q}$, we emphasize that, with respect to a pseudo-stopping time that is defined with respect to the fixed probability measure $\mathbb{Q}$, the additional flexibility of invariance times lies in the possibility to consider the martingale property under a changed measure $\mathbb{P}$.
- In fact, the pseudo-stopping time condition is very restrictive. By contrast Theorem 3 shows that invariance times are the rule rather the exception.

Example 2 (An invariance time intersecting $\mathbb{F}$ stopping times..)

- For $i = 1, 2$, let $\mu_i > 0$ be a finite $\mathbb{F}$ stopping time with bounded compensator $\mathbf{v}_i$. Assuming $\mu_2 > T$, define $\theta = \mathbb{1}_A \mu_1 + \mathbb{1}_{A^c} \mu_2$, which intersects the $\mathbb{F}$ stopping times μ_i, for some $A \in \mathcal{G}_\infty$ independent from $\mathcal{F}_\infty$ such that $\alpha = \mathbb{Q}(A) \in (0, 1)$.
- On $[0, T]$, $\mathsf{S} = \mathbb{1}_{[0,\mu_1)}\alpha + \mathbb{1}_{[0,\mu_2)}(1-\alpha)$, $\mathsf{S}_- \geq 1 - \alpha$, and $\mathbf{v} = \int_0^{\cdot\wedge\theta} \frac{1}{\mathsf{S}_{s-}} d\mathsf{D}_s \leq \frac{1}{1-\alpha}\mathsf{D}$ is bounded. Therefore the conditions of Theorem 3 are fulfilled and θ is an invariance time.
- Easy computations yield

$$\mathsf{A} = (\mathbb{1}_{[\theta,\infty)})^o = \mathbb{1}_{[\mu_1,\infty)}\alpha + \mathbb{1}_{[\mu_2,\infty)}(1-\alpha), \quad \mathsf{A}_\infty \equiv 1,$$

so that, by application of Theorem 1 (3) in Nikeghbali and Yor (2005), θ is also a pseudo-stopping time.

Example 3 (..which is not a pseudo stopping time)

- Now, to obtain an invariance time θ intersecting $\mathbb{F}$ stopping times without being a pseudo-stopping time, one can set

$$\theta = \mathbb{1}_{A_1}\mu_1 + \mathbb{1}_{A_2}\mu_2 + \mathbb{1}_{A_3}\tau,$$

  for a non pseudo-stopping time τ and a partition $A_i, i = 1, 2, 3$, independent from $\mathcal{F}_\infty$ and τ.
- With $\alpha_i = \mathbb{Q}(A_i) > 0$, we have

$$\mathsf{A} = (\mathbb{1}_{[\theta,\infty)})^o = \alpha_1 \mathbb{1}_{[\mu_1,\infty)} + \alpha_2 \mathbb{1}_{[\mu_2,\infty)} + \alpha_3 (\mathbb{1}_{[\tau,\infty)})^o,$$

  where $(\mathbb{1}_{[\tau,\infty)})^o_\infty \neq 1$, hence $\mathsf{A}_\infty \neq 1$, with positive $\mathbb{Q}$ probability. so that, by the converse part in the above mentioned theorem, θ is not a pseudo-stopping time.
- But the Azéma supermartingale of θ is given by

$$\mathsf{S} = \mathbb{1}_{[0,\mu_1]}\alpha_1 + \mathbb{1}_{[0,\mu_2]}\alpha_2 +^o (\mathbb{1}_{[0,\tau)})\alpha_3 \geq \alpha_2 \text{ on } [0, T].$$

  Hence, the other computations above do not change, which shows that θ is an invariance time.

Counterparty risk on credit derivatives

- Copula model of $\theta_0, \theta_1, \ldots, \theta_n$, where $\theta_0 = \theta$ corresponds to the default time of the counterparty of a bank in credit derivatives on names $1, \ldots, n$
- Counterparty risk computations: need make the model dynamic by introduction of a suitable model filtration $\mathbb{G}$
- Can one separate the information that comes from θ_0 from a reference filtration $\mathbb{F}$?

  $\rightarrow$ Reduction of filtration in this sense

- For applications, some kind of martingale invariance property is required, but under minimal assumptions, so that the model stays as flexible as possible in view of applications
  - → Invariance times
  - → Intensity models of counterparty risk with strong (adverse) dependence between the credit risk of a counterparty and the underlying market (credit in this case) exposure
  - → Wrong way and gap risk modeling

Dynamic copula models

- Dynamic Marshall-Olkin copula (common-shock) model
  - "Gap risk"
  - $\theta = \theta_0$ is an invariance time
  - (A) achieved with $\mathbb{P} = \mathbb{Q}$ but $\mathbb{G}$ greater than the classical progressive enlargement of $\mathbb{F}$ by θ
- Dynamic Gaussian copula model with correlation parameter $\varrho \in [0, 1]$
  - "Wrong-way risk"
  - Gives together an example of a model where the invariance property is satisfied but immersion does not hold (i.e. $\mathbb{P} \neq \mathbb{Q}$), for small ϱ
  - And, for larger ϱ, an example of a model where the invariance property may not be satisfied

Intensity Based Pricing Formulas, Survival Measure and Invariance Times: Discussion in a univariate DGC Setup

- Let

$$\theta = h^{-1}\Big(\int_0^{+\infty} \varsigma(u) dB_u \Big), \tag{8}$$

where

- ς is a Borel function on $\mathbb{R}_+$ such that $\int_0^{+\infty} \varsigma^2(u) du = 1$,
- B is a standard Brownian motion,
- $h = \Phi^{-1} \circ \mathcal{E}_\lambda$,
    - or any continuously differentiable increasing function from $(0, +\infty)$ to $\mathbb{R}$.

- The reference filtration $\mathbb{F}$ is taken as the augmented filtration of the natural filtration $\mathbb{B}$ of B.
- The full model filtration $\mathbb{G}$ is given as the augmented filtration of the progressive enlargement of $\mathbb{F} = \mathbb{B}$ by θ.
- Note that the $\mathbb{G}$ stopping time θ is $\mathcal{F}_\infty$ measurable.

- Let $h_t = \mathbb{1}_{\{\theta \leq t\}}$ and

$$m_t = \int_0^t \varsigma(u) dB_u \,, \;\; k_t = (h_t, \theta \wedge t)\,, \;\; \nu^2(t) = \int_t^{+\infty} \varsigma^2(v) dv, \tag{9}$$

with ν assumed positive for all t.

Lemma 4

The process (m, k) is $(\mathbb{G}, \mathbb{Q})$ Markov.

Lemma 5

We have

$$\mathbb{Q}(\theta > t \,|\, \mathcal{F}_t) = \Phi\Big(\frac{h(t) - m_t}{\nu(t)}\Big), \quad t \in \mathbb{R}_+, \tag{10}$$

where Φ *denotes the standard normal cdf.*

- Infinite variation $\rightarrow$ the reference filtration $\mathbb{F} = \mathbb{B}$ is not immersed into the full model filtration $\mathbb{G}$.

Lemma 6

There exists processes of the form

$$\mu_t = \mu(t, m_t, k_t) \text{ and } \gamma_t = \gamma(t, m_t, k_t) = \gamma_t \mathbb{1}_{(0,\theta]}, \quad t \in \mathbb{R}_+, \tag{11}$$

for continuous functions μ and γ with linear growth in m, such that

$$\begin{aligned} &dW_t = dB_t - \mu_t dt \text{ is a } (\mathbb{G}, \mathbb{Q}) \text{ Brownian motion and} \\ &\gamma \text{ is the } (\mathbb{G}, \mathbb{Q}) \text{ intensity of } \theta. \end{aligned} \tag{12}$$

Proposition 3

Let a process $m^\star$ satisfy

$$dm_t^\star = \varsigma(t)\big(dW_t^\star + \beta(t, m_t^\star, (0,t))dt\big)\ , \ \ 0 \le t \le T, \tag{13}$$

starting from $m_0^\star = 0$, for some Brownian motion $W^\star$ with respect to some stochastic basis $(\mathbb{G}^\star, \mathbb{Q}^\star)$. Denoting the $\mathbb{Q}^\star$ expectation by $\mathbb{E}^\star$, we have, for any positive constant T and bounded Borel function $G(t, m)$,

$$\mathbb{E}\big[\mathbb{1}_{\theta<T}G(\theta, m_\theta)\big] = \mathbb{E}^\star\Big[\int_0^T e^{-\int_0^t \gamma(s,m_s^\star,(0,s)))ds}\gamma(t, m_t^\star, (0,t))G(t, m_t^\star)dt\Big]. \tag{14}$$

Proof. by Feynman–Kac representations of PDE solutions. ∎

A contrario, we expect that

$$\mathbb{E}\big[\mathbb{1}_{\theta<T} G(\theta, m_\theta)\big] \neq \mathbb{E}\Big[\int_0^T e^{-\int_0^t \gamma(s,m_s,(0,s))ds} \gamma(t, m_t, (0,t)) G(t, m_t) dt\Big] \quad (15)$$

(except in special cases such as $G = 0$), because, from (9) and (11)–(12), it holds

$$dm_t = \varsigma(t)\big(dW_t + \beta(t, m_t, k_t)dt\big)\,, \quad t \in \mathbb{R}_+, \quad (16)$$

which, for $t \geq \theta$ so that $k_t = (1, \theta)$, diverges from the specification (13).

- In fact, let

$$V_t = \mathbb{E}\Big[\int_t^T e^{-\int_t^s \gamma(u,m_u,(0,u))du}\gamma(s,m_s,(0,s))G(s,m_s)ds\,\Big|\,\mathcal{G}_t\Big],\ \ t\in\mathbb{R}_+,$$

so that V_0 is equal to the right hand side in (15).
- By an application of Duffie et al. (1996, Proposition 1) with $X=r=0$ and $h_\cdot=\gamma(\cdot,m_\cdot,(0,\cdot))$ on $[0,T]$ there, we have

$$\mathbb{E}\big[\mathbb{1}_{\theta<T}G(\theta,m_\theta)\big] = V_0 - \mathbb{E}\Delta_\theta V. \tag{17}$$

    - Any process coinciding with the $(\mathbb{G},\mathbb{Q})$ intensity of θ before θ is an eligible process h in their setup

- In a basic immersed setup, $\mathbb{E}\Delta_\theta V$ vanishes and equality holds in (15): see the comments before Section 3 in Duffie et al. (1996), page 1379 in Collin-Dufresne et al. (2004), or following (3.22), (H.3) and Proposition 6.1 in Bielecki and Rutkowski (2001)).
- But, in general, $\mathbb{E}\Delta_\theta V$ is nonnull and intractable.

- One specific instance of (13), which corresponds to the approach by Collin-Dufresne et al. (2004, Theorem 1), consists in using $m^* = m$ and $W^* = W$, taking for $\mathbb{Q}^*$ the so-called survival measure[1] with $(\mathbb{G}, \mathbb{Q})$ density process $e^{\int_0^\cdot \gamma(u, m_u, (0,u))du} \mathbb{1}_{(0,\theta]}$.
- This fixes the discrepancy in (15) by singularly changing the probability measure, while sticking to the original model filtration $\mathbb{G}$
    - or, more precisely, resorting to the $\mathbb{Q}^*$ augmentation $\mathbb{G}^*$ of $\mathbb{G}$, obtained by adding to $\mathbb{G}$ all the $\mathbb{Q}^*$ null sets A such that $A \subseteq \{\theta \leq T\}$).

[1] The "survival measure" idea and terminology were first introduced and used for various purposes in Schönbucher (1999, 2004)).

- Another instance of (13) consists in using

$$m^* = m \text{ and } dW^*_t = dB_t - \mu(t, m_t, (0,t))dt.$$

- As it follows from Lemma 3.5 and Section 4.4 in Crépey and Song (2017), this process W^* is a $(\mathbb{G}^* = \mathbb{F} = \mathbb{B}, \mathbb{Q}^* = \mathbb{P})$ Brownian motion, for some probability measure $\mathbb{P}$, distinct from $\mathbb{Q}$ but equivalent to it on $\mathcal{F}_T$, on which $\mathbb{P}$ is uniquely determined through (6).

- The corresponding formula (14) is none other than our expectation formula (7).
- This approach fixes the discrepancy in (15) by reducing the filtration from $\mathbb{G}$ to $\mathbb{F} = \mathbb{B}$, while changing the probability measure “as little as possible”, i.e. equivalently on $\mathcal{F}_T$
  - In a basic immersive setup, an invariance time approach does not change $\mathbb{Q}$ at all, whereas Collin-Dufresne et al. (2004)'s measure change is still singular

- We emphasize that Collin-Dufresne et al. (2004)'s approach only provides a transfer of conditional expectation formulas (because of the singularity of their measure change), as opposed to a transfer of semimartingale calculus as a whole under an invariance time approach.
- From a more general and theoretical perspective, these different approaches can be related via the generalized Girsanov formulas of Kunita (1976) and Yoeurp (1985) (cf. Song (2013)).

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