Point processes in random environment and application to the study of longevity risk

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Some application of point processes

- Renewal of interest in **point processes** in the past years.
- Flexibility allows for the modeling of a wide range of phenomena in:
    - Finance and insurance, neurosciences, biology and ecology, biochemical systems, epidemiology, cyber risk..
- **Human longevity**
    - Point processes appear in the study of population dynamics: naturally in complex random environment.
    - Impact of heterogeneity on longevity indicators

Kaakaï, S. and El Karoui, Nicole. Birth Death Swap population in random environment and aggregation with two timescales, arXiv:1803.00790, 2020.

Outline

Stochastic intensity

- An adapted counting process N, admits the $(\mathcal{G}_t)$-(predictable) stochastic intensity (λ_t) if

$$N_t - \int_0^t \lambda_s \mathrm{d}s \text{ is a } (\mathcal{G}_t)\text{- local martingale.}$$

- $\mathsf{P}(N_{t+dt} - N_t = 1|\mathcal{G}_t) \simeq \lambda_t \mathrm{d}t.$

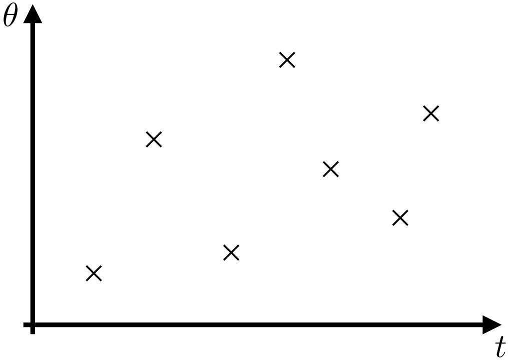
- Equivalently, for all nonnegative predictable processes C

$$\mathsf{E}[\int_0^\infty C_s \mathrm{d}N_s] = \mathsf{E}[\int_0^\infty C_s \lambda_s \mathrm{d}s].$$

- In general:
  - (λ_t), does not characterize the distribution of N.
  - λ_t is written as a functional of N

$$\lambda_t(\omega) = \alpha(\omega, t, [N]_{t-}).$$

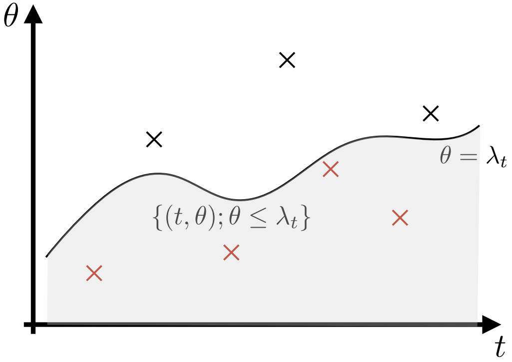
Pathwise representation



Space-time $(\mathcal{G}_t)$ Poisson measure Q on $\mathbb{R}^+ \times \mathbb{R}^+$ of mean measure $\mathrm{d}t \otimes \mathrm{d}\theta$.

Pathwise representation

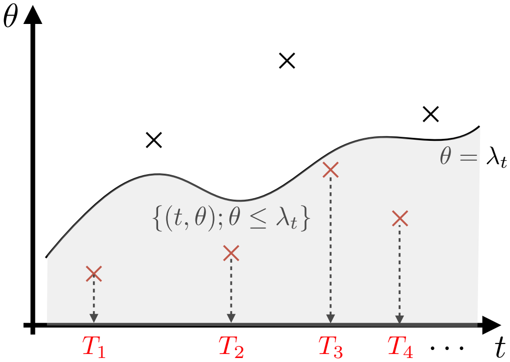
Given a predictable process $(\lambda_t)_{t\geqslant 0}$ with $\int_0^t \lambda_s \mathrm{d}s < +\infty$ a.s. $\forall t \geqslant 0$ (nonexplosion condition)



Restriction to predictable subset:

$$\{(s,\theta);\ \theta \leqslant \lambda_s(\omega),\ s \leqslant t\}$$

Pathwise representation



Thinning equation

$N_t^\lambda = \int_0^t \int_{\mathbb{R}^+} \mathbb{1}_{\{\theta \leqslant \lambda_s\}} Q(\mathrm{d}s, \mathrm{d}\theta)$ is a *counting process* of $(\mathcal{G}_t)$-intensity λ_t.

SDEs driven by Poisson measures

- When $\lambda_t = \alpha(\omega, t, [N]_{t-})$ is a functional of N, thinning equation $\Rightarrow$ SDE driven by Q:

$$N_t^\alpha = \int_0^t \int_{\mathbb{R}^+} \mathbb{1}_{\{\theta \leqslant \alpha(s,[N^\alpha]_{s-})\}} Q(\mathrm{d}s, \mathrm{d}\theta), \quad dN_t^\alpha = Q(\mathrm{d}t,]0, \alpha(t, [N^\alpha]_{t-})]) \tag{1}$$

- **Existence of a well-defined (non-exploding) solution?**
- Yes if α is "strongly" majorized by a "good" function β: $\alpha \leqslant_s \beta$

$$\forall t \geqslant 0, \sup_{[m] \prec [n]} \alpha(t, [m]) \leqslant \beta(t, [n]) \; a.s.$$

Strong comparison

$$N_t^\alpha = \int_0^t \int_{\mathbb{R}^+} \mathbb{1}_{\{\theta \leqslant \alpha(s,[N^\alpha]_{s-})\}} Q(\mathrm{d}s, \mathrm{d}\theta), \quad dN_t^\alpha = Q(\mathrm{d}t,]0, \alpha(t, [N^\alpha]_{t-})]) \quad (2)$$

Results

- If $\alpha \leqslant_s \beta$ with
$$\beta(t,[n]) = k_t g(n(t))$$
(k_t) predictable locally bounded process and g verifying $\sum \frac{1}{g(j)} = \infty$, then (1) admits a unique well-defined solution N^α.
- Furthermore, N^α is **strongly dominated** by the counting process N^β of intensity functional β and obtained with the same Poisson measure : $N^\alpha \prec N^\beta$), i.e.
$N^\beta - N^\alpha$ is a counting process (jump times of N^α = jump times of N^β).

Converse result

Let N^λ with $(\mathcal{G}_t)$- intensity (λ_t) and such that $N^\lambda \prec N$.
Is there a representation of N^λ in terms of stochastic integral with respect to a marked process with same jumps than N?

- **YES**

Corollary

Two counting processes with the same intensity functional and strongly dominated by the same process have the same distribution.

Outline

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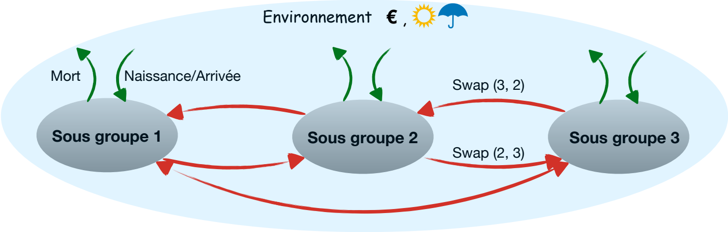
The model

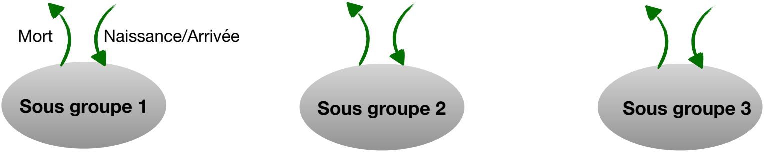


- Population process $Z = (Z^i)_{i=1..p}$ structured by discrete subgroups adapted to a history $(\mathcal{G}_t) \supset (\mathcal{F}_t^Z)$.
- Population evolves according to demographic events (births/arrival, death/exit) or changes of characteristics (swap).
- Random environment $\Rightarrow$ stochastic event intensities:

$$\mathrm{P}(\text{ ev of type } \gamma \in]t, t + \mathrm{d}t] | \mathcal{G}_t) \simeq \mu^\gamma(\omega, t, Z_t)\mathrm{d}t.$$

Standard framework

Markov multi-type Birth-Death processes



1. Only demographic events.
2. Birth and death intensity only depend on the state of the population.

$$\mathrm{P}(\text{ ev of type } \gamma \in]t, t+\mathrm{d}t] | \mathcal{G}_t) \simeq \mu^\gamma(Z_t)\mathrm{d}t.$$

Example: effect of habitat fragmentation

(from Pichancourt et al (2006))

- Population evolving on different type of habitats (favorable and unfavorable)
- $\nearrow$ of habitat fragmentation $\Rightarrow$ $\nearrow$ migration between patches $\Rightarrow$ $\nearrow$ probability of being in unfavorable habitat.
- Effect of environment: e.g. weather, habitat transformation, human control,...

Example 2 : Botnets interactations

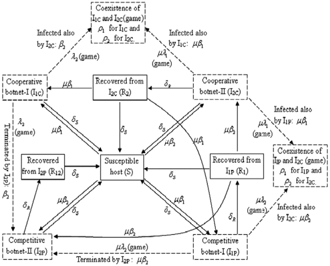
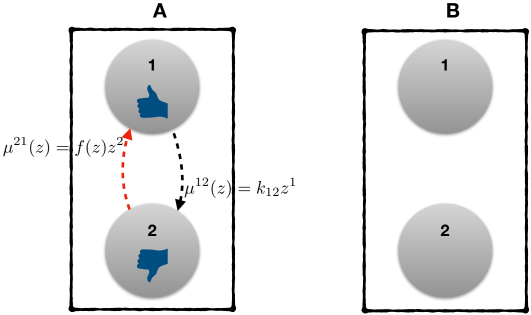


Figure: From Song, Jin and Sun (2011).

- Botnet : network of thousands of computers under the control of a botnet owner. ↳ One of the most serious cyber risk.
- Botnet owners try to increase the size of their botnets to survive.
- Market saturation ⇒ interactions between botnets owner.
- Two strategies: cooperation and competition.

A toy example



- Compositional effects (Dowd(2014)).

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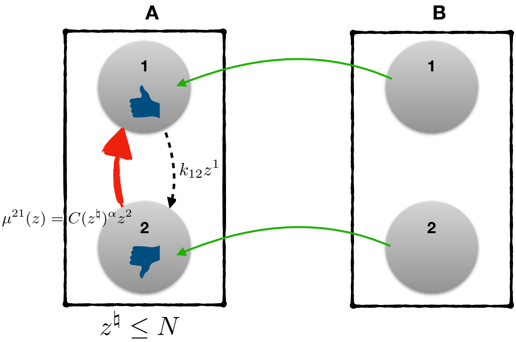

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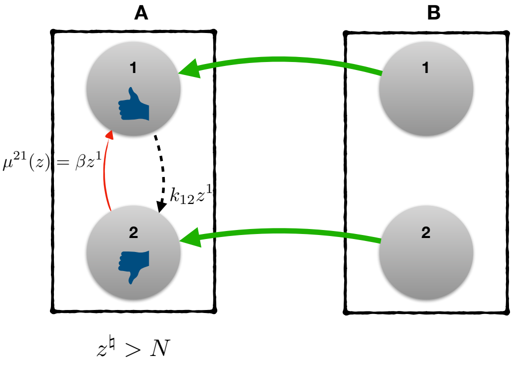

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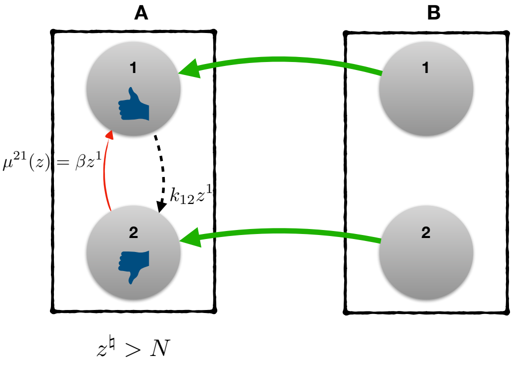
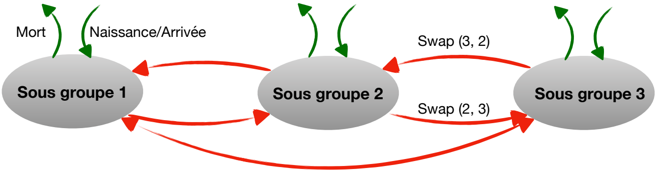


- Compositional effects (Dowd(2014)).
- Individuals marked by numerous characteristics...
- ....

Events description



- **Population process** structured in p subgroups: population process $(Z_t) = ((Z_t^i)_{i=1}^p)$ counting the number of individuals in each subgroup.
- **Events description** $p(p+1)$ types of events $\gamma \in \mathcal{J}$.
  - **Birth events** in subgroup j: $\Delta Z_t = \mathbf{e}_j = (0, \dots, 1_j, 0, \dots)$.
  - **Death events** in subgroup i: $\Delta Z_t = -\mathbf{e}_i = (0, \dots, -1_i, 0, \dots)$.
  - **Swap events** from subgroup i to j:
    $\Delta Z_t = \mathbf{e}_j - \mathbf{e}_i = (0, .., 0, -1_i, 0, .., 1_j, 0, \dots)$.

Link with point processes

Idea Represent the population with point processes.

- Each type of event (birth, death, swap) $\gamma \in \mathcal{J}$ is associated with the counting process:

$$N_t^\gamma = \sum_{0<s\leqslant t} \mathbb{1}_{\{\Delta Z_s = \phi(\gamma)\}} \tag{3}$$

- $p(p+1)$ multivariate counting process $\mathbf{N} = (N^\gamma)_{\gamma\in\mathcal{J}}$.

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- $p(p+1)$ multivariate counting process $\mathbf{N} = (N^\gamma)_{\gamma \in \mathcal{J}}$.

- The population process can be expressed as a linear function of $\mathbf{N}$:
$$Z_t^k = Z_0^k + N_t^{b,k} - N_t^{d,k} + \sum_{j \neq k} N_t^{jk} - \sum_{i \neq k} N_t^{ki} \quad \forall k = 1..p.$$
- Vector notation:
$$Z_t = Z_0 + \mathbf{N}_t^b - \mathbf{N}_t^d + \phi^s \odot \mathbf{N}_t^s \in \mathbb{N}^p. \tag{4}$$

with $\mathbf{N}_t^b \in \mathbb{N}^p, \quad \mathbf{N}_t^d \in \mathbb{N}^p, \quad \mathbf{N}_t^s = (N_t^{ij})_{\substack{i,j=1..p \\ i \neq j}} \in \mathbb{N}^{p(p-1)}$.

Stochastic intensities

The BDS process is formally defined through its events counting process **N**.

- **Ingredient 1** : an intensity functional $\boldsymbol{\mu} = (\mu^\gamma)_{\gamma \in \mathcal{J}}$.
- $\forall \gamma \in \mathcal{J}$, N^γ has the $\mathcal{G}_t$- (predictable) intensity $\boldsymbol{\mu}(\omega, t, Z_{t-})$:

$$P(N^\gamma_{t+dt} - N^\gamma_t = 1 | \mathcal{G}_t) \simeq \mu^\gamma(t, Z_t) dt$$

  - $N^\gamma_t - \int_0^t \mu^\gamma(s, Z_s) \mathrm{d}s$ is a $\mathcal{G}_t$-local martingale.
  - Support condition (no death or swap from an empty class):

$$\mu^{i\beta}(t, z) \mathbf{1}_{\{z^i = 0\}} \equiv 0 \quad \forall i \in \mathcal{J}_p,\ \beta \in \mathcal{J}^{(i)}.$$

Examples

- Poisson process: $\mu^\gamma \equiv c^\gamma$.

$$\mathrm{E}[N_t^\gamma] = c^\gamma t.$$

- Linear birth intensity:

$$\mu^{b,i}(\omega, t, z) = b_t^i(\omega) z^i + \underbrace{\lambda^i(t, Y_t)}_{\text{entry rate}}.$$

- Death intensity :

$$\mu^{d,i}(\omega, t, z) = d_t^i(\omega) z^i + \sum_{j=1}^{p} \underbrace{c(z^i, z^j)}_{\text{competition}}.$$

- Extension to path dependent intensity functionals.

BDS process SDE

BDS process is formally defined through its events counting process **N**.

- **Ingredient 1**: an intensity functional $\boldsymbol{\mu} = (\mu^\gamma)_\gamma$.
- **Ingredient 2** : Thinning and projection of space-time Poisson measure.
  - **Driving multivariate Poisson measures** family of (p+1)p independent space-time Poisson measures $\mathbf{Q}(ds, d\theta) = (Q^\gamma(ds, d\theta))_{\gamma \in \mathcal{J}}$ on $\mathbb{R}^+ \times \mathbb{R}^+$ (intensity $\mathrm{d}t \otimes \mathrm{d}\theta$).

$$N_t^\gamma = \int_0^t \int_{\mathbb{R}^+} \mathbb{1}_{\{\theta \leqslant \mu^\gamma(s, Z_{s-})\}} Q^\gamma(\mathrm{d}s, \mathrm{d}\theta), \quad \forall \gamma \in \mathcal{J}.$$

  - **Birth Death Swap SDE**:

$$\mathbf{N}_t = \int_0^t \int_{\mathbb{R}^+} \mathbb{1}_{\{\theta \leqslant \boldsymbol{\mu}(s, Z_{s-})\}} \mathbf{Q}(\mathrm{d}s, \mathrm{d}\theta), \quad Z_t = F(Z_0, \mathbf{N}_t). \quad (5)$$

BDS process SDE

Existence of non-explosive solutions: control birth intensities

$$\mu^b(\omega, t, z) \leqslant k_t \mathbf{g}(z^\natural) = \mathbf{g}(\sum_{i=1}^{p} z^i), \qquad (6)$$

with $\mathbf{g}$ verifying $\sum_{n\geqslant 1} \frac{1}{\sum g^i(n)} = \infty$.

Proposition (K., El Karoui)

There exists a unique well-defined solution $\mathbf{N}$ *of* (5), **strongly dominated** *by a multivariate counting process* $\mathbf{G}$*:* $\mathbf{G} - \mathbf{N}$ *is a multivariate counting process.*

The triplet $(Z_0, \mathbf{N}, Z)$ *defines a Birth Death Swap process of intensity functional* $\boldsymbol{\mu}$ *and driven by* $\mathbf{Q}$.

Outline

Population with two time-scales

Study of the population evolution when composition changes occurs at a fast pace in comparison with the demographic scale.

- **Hyp**: intensity of swap events $>>$ demographic events
- BDS process with intensity function $\boldsymbol{\mu}^{\epsilon} = (\boldsymbol{\mu}^{\mathrm{dem}}, \frac{1}{\epsilon}\boldsymbol{\mu}^{s})$:

$$\mathrm{d}\mathbf{N}_t^{s,\epsilon} = \mathbf{Q}^s(\mathrm{d}t, [0, \frac{1}{\epsilon}\boldsymbol{\mu}^s(t, Z^{\epsilon}_{t-})]), \quad \mathrm{d}\mathbf{N}_t^{\mathrm{dem},\epsilon} = \mathbf{Q}^{\mathrm{dem}}(\mathrm{d}t, [0, \boldsymbol{\mu}^{\mathrm{dem}}(t, Z^{\epsilon}_{t-})]).$$

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- $\mathbf{N}^{s,\epsilon}$: explosion when $\epsilon \to 0$.
- The demographic intensity functional $\boldsymbol{\mu}^{\mathrm{dem}}$ is not modified $\Rightarrow$ uniform strong domination of $(\mathbf{N}^{\mathrm{dem},\epsilon})$

$$\forall \epsilon > 0, \quad \mathbf{N}^{\mathrm{dem},\epsilon} \prec \mathbf{G}^{\mathrm{dem}}.$$

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$$\forall \epsilon > 0, \quad \mathbf{N}^{\mathrm{dem},\epsilon} \prec \mathbf{G}^{\mathrm{dem}}.$$

Consequence: $(\mathbf{N}^{\mathrm{dem},\epsilon})_{\epsilon}$ is tight in $\mathcal{A}^{2p}$ (space of multivariate counting functions).

Aggregated process

Goal Study limit points of $(\mathbf{N}^{\mathrm{dem},\epsilon})$.

- Example of application Study of the "macro population"

$$Z_t^{\natural,\epsilon} = \sum_{i=1}^{p} Z_t^{i,\epsilon},$$

with aggregated birth and death intensities:

$$\mu^{b,\natural}(t, Z_t^{\epsilon}) = \sum_{i=1}^{p} \mu^{b,i}(t, Z_t), \quad \mu^{d,\natural}(t, Z_t^{\epsilon}) = \sum_{i=1}^{p} \mu^{d,i}(t, Z_t^{\epsilon})$$

- Population viability? Impact of composition changes on aggregated demographic rates?
- **Difficulty**: Not a "true" Birth-Death process.
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$$Z_t^{\natural,\epsilon} = \sum_{i=1}^{p} Z_t^{i,\epsilon} = Z_0 + \sum_{i=1}^{p} \left(N_t^{b,i,\epsilon} - N_t^{d,i,\epsilon} \right) = F(Z_0, \mathbf{N}_t^{\mathrm{dem},\epsilon}),$$

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Identification of limit points of $(\mathbf{N}^{\mathrm{dem},\epsilon})$

- Natural framework: study of $\mathcal{G}_t$-local martingales $\mathbf{N}_t^{\mathrm{dem},\epsilon} - \int_0^t \boldsymbol{\mu}^{\mathrm{dem}}(\omega, s, Z_{s-}^{\epsilon})\mathrm{d}s$.
- Deterministic intensity functional (Markov framework) $\Rightarrow$ Averaging result of Kurtz (1992).
- Here: $\boldsymbol{\mu}^{\mathrm{dem}}(\omega, t, z)$ + intensity functional does not characterize the distribution of $\mathbf{N}^{\mathrm{dem},\epsilon}$.

Need convergence of random functionals preserving probabilistic structure $\Rightarrow$ **Stable convergence**.

Outline

Stable convergence

- Originated by Alfred Rényi
- (Partial) references: Aldous et al. (1978), Jacod and Memin (1981), Hausler and Luschgy (2015).

Convergence in distribution and stable convergence

- Let $(Y_n)_{n\geqslant 0}$ be a sequence of $(E,\mathcal{E})$-valued random variables, with μ^n the distribution Y_n and μ the distribution of Y.
- (Y_n) **converges to Y in distribution** (weakly) iff for all bounded continuous functions $f \in C_{bc}(E)$,

$$\mu^n(f) = \int_E f(x)\mu^n(\mathrm{d}x) \to_{n\to\infty} \mu(f) = \int_E f(x)\mu(\mathrm{d}x).$$

- Equivalently

$$\forall f \in C_{bc}(E), \quad \mathrm{E}[f(Y_n)] = \mu^n(f) \to \mathrm{E}[f(Y)] = \mu(f).$$

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$$\mathrm{E}[\cancel{f}(Y_n)] \to \hat{\mathrm{E}}[\cancel{f}(Y)].$$

$$\mathrm{E}[H(\cdot, Y_n)] \to ?$$

Space of rules

- **Class of test functions** $C_{bmc}(\Omega \times E)$: Bounded measurable functions $H : \Omega \times E \to \mathbb{R}$, with $H(\omega, \cdot)$ continuous.
- **Idea** write

$$\mathsf{E}[H(\cdot, Y_n)] = \mathsf{R}^n(H) = \int_{\Omega \times \mathsf{E}} H(\omega, x)\mathsf{R}^n(\mathrm{d}\omega, \mathrm{d}x).$$

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- Take: $\mathsf{R}^n(\mathrm{d}\omega, \mathrm{d}x) = \mathsf{P}(\mathrm{d}\omega)\delta_{Y_n}(\omega)(\mathrm{d}x)$.

Space of rules

- Probability measures R on $\Omega \times E$ with marginal P on Ω.
- Disintegration $R(\mathrm{d}\omega, \mathrm{d}x) = \mathsf{P}(\mathrm{d}\omega)\Gamma(\omega, \mathrm{d}x)$

$$R(H) = \int_{\Omega} \mathsf{P}(\mathrm{d}\omega) \underbrace{\int_E H(\omega, x)\Gamma(\omega, \mathrm{d}x)}_{\Gamma(H)} = \mathsf{E}[\Gamma(H)]$$

Stable convergence of random variables

Stable convergence of (Y_n) to a rule R:

- Convergence of probability measures on the space of rules:

$$R^n(H) \to R(H),\ \forall\ H \in C_{bmc}(\Omega \times E).$$

- 2 interpretations:

1. Convergence of the given space:

$$\text{View 1} \qquad R^n(H) = \mathrm{E}[H(Y_n)] \to \mathrm{E}[\Gamma(H)]\ \ (= R(H)).$$

2. Convergence to an r.v on extended space $(\Omega \times E, \bar{\mathcal{G}}, \mathrm{R})$ with $\bar{Y}(\omega, x) = x$:

$$\text{View 2} \qquad \mathrm{E}[H(Y^n)] \to \mathrm{R}[H(\bar{Y})].$$

Properties

- Mode of convergence stronger than convergence in distribution.
- **Property** (Jacod and Memin (1981)

  If (Yn) (μ^n) converges in distribution to Y (μ), there exists a subsequence of (Yn) converging stably to a rule R.
- In particular, if (μ^n) is tight, then there exists a subsequence of (Yn) converging stably to a rule R.

Agenda Apply stable convergence to obtain averaging results for point processes with stochastic intensities.

Outline

Two timescales BDS processes

- Two timescales BDS system:

$$\mathrm{d}\mathbf{N}_t^{s,\epsilon} = \mathbf{Q}^s(\mathrm{d}t, [0, \frac{1}{\epsilon}\boldsymbol{\mu}^s(t, Z_{t-}^\epsilon)]), \quad \mathrm{d}\mathbf{N}_t^{\mathrm{dem},\epsilon} = \mathbf{Q}^{\mathrm{dem}}(\mathrm{d}t, [0, \boldsymbol{\mu}^{\mathrm{dem}}(t, Z_{t-}^\epsilon)]).$$

$$Z_t^\epsilon = Z_0 + \mathbf{N}_t^{b,\epsilon} + \mathbf{N}_t^{d,\epsilon} + \phi^s \odot \mathbf{N}_t^{s,\epsilon}$$

- **Variable of interest**: $2p$-multivariate counting $\mathbf{N}^{\mathrm{dem},\epsilon}$.
- State space: $E = \mathcal{A}^{2p}$
  - Subspace of Skorohod space $D(\mathbb{R}^+, \mathbb{N}^{2p})$ of counting functions.
  - $\mathcal{F}_t^{\mathcal{A}} = \sigma(\alpha(s); s \leqslant t,\ \alpha \in \mathcal{A}^{2p})$.

Stable limits of demographic process

- $(\mathbf{N}^{\mathrm{dem},\epsilon})$ is tight in $\mathcal{A}^{2p} \Rightarrow$ subsequence converging stably.
- **Enlarged space**: $(\Omega \times E, (\bar{\mathcal{G}}_t)) = (\Omega \times \mathcal{A}^{2p}, (\mathcal{G}_t \otimes \mathcal{F}_t^{\mathcal{A}}))$.

$$\text{Stable } \bar{\mathbf{N}}^{\mathrm{dem}}(\omega, \alpha) = \alpha \in \mathcal{A}^{2p}.$$

- $(\mathbf{N}^{\mathrm{dem},\epsilon})$ converges stably to $\bar{\mathbf{N}}^{\mathrm{dem}}$ on $(\Omega \times \mathcal{A}^{2p}, (\mathcal{G}_t \otimes \mathcal{F}_t^{\mathcal{A}}), \mathrm{R}^{\mathrm{dem}})$ if

$$\mathrm{E}[H(\mathbf{N}^{\mathrm{dem},\epsilon})] \underset{\epsilon \to 0}{\longrightarrow} \mathrm{R}^{\mathrm{dem}}[H(\bar{\mathbf{N}}^{\mathrm{dem}})], \quad \forall H \in \mathcal{C}_{bmc}(\Omega \times \mathcal{A}^{2p}).$$

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$$\mathrm{E}[\mathbb{1}_B f(\mathbf{N}^{\mathrm{dem},\epsilon})] \underset{\epsilon \to 0}{\longrightarrow} \mathrm{R}^{\mathrm{dem}}[\mathbb{1}_B f(\bar{\mathbf{N}}^{\mathrm{dem}})], \quad \forall\, B \in \mathcal{G},\ f \in \mathcal{C}_{cb}(\mathcal{A}^{2p}).$$

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- **A first property** Conservation of strong domination at the limit

$$\bar{\mathbf{N}}^{\mathrm{dem}} \prec \mathbf{G}^{\mathrm{dem}}, \quad \mathrm{R}^{\mathrm{dem}} \text{ a.s.}$$

$(\mathbf{G}^{\mathrm{dem}}(\omega, s) = \mathbf{G}^{\mathrm{dem}}(\omega).)$

Limit compensator

Second step: Study of the limit compensators.

- $\mathbf{N}^{\mathrm{dem},\epsilon}$ have for $(\mathcal{G}_t)$-compensator:

$$\mathbf{A}^{\epsilon} = \int_0^{\cdot} \boldsymbol{\mu}^{\mathrm{dem}}(\omega, s, Z_s^{\epsilon})\mathrm{d}s.$$

- **Issue** Family of population processes $(Z^{\epsilon}) = (g(Z_0, \mathbf{N}^{\mathrm{dem},\epsilon}, \mathbf{N}^{s,\epsilon}))$ is not tight, due to explosion of swap events.

Limit compensator (II)

- $\mathbf{N}^{\mathrm{dem},\epsilon}$ have for $(\mathcal{G}_t)$-compensator:

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- $(Z^{\epsilon}) = (g(Z_0, \mathbf{N}^{\mathrm{dem},\epsilon}, \mathbf{N}^{s,\epsilon}))$ is not tight.

Actually, we are interested in convergence of quantities $\mathsf{E}[\int_0^t \boldsymbol{\lambda}(s, Z_s^{\epsilon})\mathrm{d}s]$.

- **Idea** See Z^{ϵ} as an $\mathbb{N}^p$-valued random variable on $\Omega \times \mathbb{R}^+$

$$\tilde{Z}^{\epsilon}(\omega, s) = Z_s^{\epsilon}(\omega), \quad \tilde{E}[\boldsymbol{\lambda}(\cdot, \tilde{Z}^{\epsilon})] = \mathsf{E}[\int \boldsymbol{\lambda}(\cdot, \tilde{Z}_s^{\epsilon})\mathrm{d}s].$$

- Stable limits of $\tilde{Z}^{\epsilon}$ with view 1 : random kernels

$$\Gamma(\omega, s, \mathrm{d}z)$$

Limit compensator (II)

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- **Joint** Stable limits of $\tilde{Z}^{\epsilon}$ with view 1 : random kernels

$$\Gamma(\omega, s, [\bar{\mathbf{N}}^{\mathrm{dem}}]_s, \mathrm{d}z)$$

$$\mathrm{E}[\int_0^t \boldsymbol{\lambda}(s, Z_s^{\epsilon})\mathrm{d}s] \to_{\epsilon} \mathrm{R}^{\mathrm{dem}}[\int_0^t \int_{\mathbb{N}^p} \boldsymbol{\lambda}(s, z)\Gamma_s([\bar{\mathbf{N}}^{\mathrm{dem}}]_s, \mathrm{d}z)\mathrm{d}s]$$

Summary/General averaging result

Stable limits of $(\mathbf{N}^{\mathrm{dem},\epsilon})$ are multivariate counting processes:

1. Defined on an extension $(\Omega \times \mathcal{A}^{2p}, (\mathcal{G}_t \otimes \mathcal{F}_t^{\mathcal{A}}), \mathrm{R}^{\mathrm{dem}})$ of $(\Omega, (\mathcal{G}_t), \mathrm{P})$.
2. $\mathbf{N}^{\mathrm{dem},\epsilon} \prec \mathbf{G}^{\mathrm{dem}}$.
3. **Limit demographic intensity**

$$\bar{\mathbf{N}}^{\mathrm{dem}} \text{ has the } (\bar{\mathcal{G}}_t)\text{-intensity } (\Gamma_s[\bar{\mathbf{N}}^{\mathrm{dem}}]_s, \boldsymbol{\mu}^{\mathrm{dem}}) = \int_{\mathbb{N}^p} \boldsymbol{\mu}^{\mathrm{dem}}(s,z)\Gamma_s[\bar{\mathbf{N}}^{\mathrm{dem}}]_s, \mathrm{d}z).$$

- At the limit, the demographic intensity is averaged against stable limits of the population variables $(\tilde{Z}^{\epsilon})$.

Properties of averaging kernels (I)

- Let $f \in \mathcal{C}_b(\mathbb{N}^p)$. $(f(Z_t^\epsilon))_t$ is a pure jump process.
- Jump of type γ occurs $\Rightarrow$ jump $f(Z_{t-}^\epsilon + \phi(\gamma)) - f(Z_{t-}^\epsilon)$, so that:

$$
\begin{aligned}
f(Z_t^\epsilon) - f(Z_0) \\
&= \sum_{\gamma \in \mathcal{J}^{\mathrm{dem}}} \int_0^t \big(f(Z_{s-}^\epsilon + \phi(\gamma)) - f(Z_{s-}^\epsilon) \big) \mathrm{d}N_s^{\gamma,\epsilon} \\
&+ \sum_{\gamma \in \mathcal{J}^{\mathrm{sw}}} \int_0^t \big(f(Z_{s-}^\epsilon + \gamma) - f(Z_{s-}^\epsilon) \big) \mathrm{d}N_s^{\gamma,\epsilon}
\end{aligned}
$$

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$$f(Z_t^\epsilon) - f(Z_0) - \sum_{\gamma \in \mathcal{J}^{\mathrm{dem}}} \int_0^t \underbrace{\big(f(Z_{s-}^\epsilon + \phi(\gamma)) - f(Z_{s-}^\epsilon)\big)\mu^\gamma(s, Z^\epsilon)}_{L_t^{\mathrm{dem}} f(Z_t^\epsilon)} \mathrm{d}s$$

$$= \sum_{\gamma \in \mathcal{J}^{\mathrm{dem}}} \int_0^t \big(f(Z_{s-}^\epsilon + \phi(\gamma)) - f(Z_{s-}^\epsilon)\big)\big(\mathrm{d}N_s^{\gamma,\epsilon} - \mu^\gamma(s, Z_s^\epsilon)\mathrm{d}s\big)$$

local martingale

$$+ \sum_{\gamma \in \mathcal{J}^{\mathrm{sw}}} \int_0^t \big(f(Z_{s-}^\epsilon + \gamma) - f(Z_{s-}^\epsilon)\big)\mathrm{d}N_s^{\gamma,\epsilon}$$

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f(Z_t^\epsilon) - f(Z_0) &- \sum_{\gamma \in \mathcal{J}^{\mathrm{dem}}} \int_0^t \underbrace{\big(f(Z_{s-}^\epsilon + \phi(\gamma)) - f(Z_{s-}^\epsilon)\big)\mu^\gamma(s, Z^\epsilon)}_{L_t^{\mathrm{dem}} f(Z_t^\epsilon)} \mathrm{d}s \\
&- \frac{1}{\epsilon} \sum_{\gamma \in \mathcal{J}^{\mathrm{sw}}} \int_0^t \underbrace{\big(f(Z_{s-}^\epsilon + \phi(\gamma)) - f(Z_{s-}^\epsilon)\big)\mu^\gamma(s, Z^\epsilon)\mathrm{d}s}_{L_t^{\mathrm{sw}} f(Z_t^\epsilon)} \\
&= \sum_{\gamma \in \mathcal{J}^{\mathrm{dem}}} \int_0^t \big(f(Z_{s-}^\epsilon + \phi(\gamma)) - f(Z_{s-}^\epsilon)\big)\big(\mathrm{d}N_s^{\gamma,\epsilon} - \mu^\gamma(s, Z_s^\epsilon)\mathrm{d}s\big) \quad \text{local martingale} \\
&+ \sum_{\gamma \in \mathcal{J}^{\mathrm{sw}}} \int_0^t \big(f(Z_{s-}^\epsilon + \gamma) - f(Z_{s-}^\epsilon)\big)\big(\mathrm{d}N_s^{\gamma,\epsilon} - \frac{1}{\epsilon}\mu^\gamma(s, Z_s^\epsilon)\mathrm{d}s\big) \quad \text{local martingale}
\end{aligned}
$$

Properties of averaging kernels (II)

$$f(Z_t^\epsilon) - f(Z_0) - \int_0^t L_s^{\mathrm{dem}} f(Z_s^\epsilon)\mathrm{d}s - \frac{1}{\epsilon}\int_0^t L_s^{\mathrm{sw}} f(Z_s^\epsilon)\mathrm{d}s \text{ is a } (\mathcal{G}_t)\text{-local martingale.}$$

Properties of averaging kernels (II)

$$\epsilon\Big(f(Z_t^\epsilon)-f(Z_0)-\int_0^t L_s^{\mathrm{dem}} f(Z_s^\epsilon)\mathrm{d}s\Big)-\int_0^t L_s^{\mathrm{sw}} f(Z_s^\epsilon)\mathrm{d}s \text{ is a } (\mathcal{G}_t)\text{-local martingale.}$$

Properties of averaging kernels (II)

$$\underbrace{\epsilon\Big(f(Z_t^\epsilon) - f(Z_0) - \int_0^t L_s^{\mathrm{dem}} f(Z_s^\epsilon)\mathrm{d}s\Big)}_{\to_\epsilon 0} - \underbrace{\int_0^t L_s^{\mathrm{sw}} f(Z_s^\epsilon)\mathrm{d}s}_{\to_\epsilon \int_0^t \Gamma_s([\bar{N}^{\mathrm{dem}}]_s, L_s^{\mathrm{sw}} f)\mathrm{d}s} \quad \text{is a } (\mathcal{G}_t)\text{-local martingale.}$$

(along a subsequence)

- **Proposition** The random kernel Γ must satisfy

$$\Gamma_s([\bar{N}^{\mathrm{dem}}]_s, L_s^{\mathrm{sw}} f) = \int_{\mathbb{N}^p} L_s^{\mathrm{sw}} f(z)\Gamma_s([\bar{N}^{\mathrm{dem}}]_s, \mathrm{d}z) = 0, \quad \mathrm{R}^{\mathrm{dem}} \otimes \mathrm{d}s, \ a.s. \ (7)$$

Link with pure swap processes

$$\Gamma_s([\bar{N}^{\mathrm{dem}}]_s, L_s^{\mathrm{sw}} f) = 0, \quad \mathrm{R}^{\mathrm{dem}} \otimes \mathrm{d}s, \ a.s., \quad (5)$$

- **Pure swap processes** S :
  - Population with NO demographic events.
  - Constant size: $S_0 \in \mathcal{U}_d \Rightarrow S_t \in \mathcal{U}_d$, populations of size d.
  - Swap random operator:

$$L_s^{\mathrm{sw}}(\omega) f(z) = \sum_{\substack{i,j=1 \\ i \neq j}}^{p} \big(f(z+\mathbf{e}_j-\mathbf{e}_i)-f(z)\big)\mu^{(i,j)}(\omega, s, z), \quad \forall s \geqslant 0, \ z \in \mathbb{N}^p,$$

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  - Constant size: $S_0 \in \mathcal{U}_d \Rightarrow S_t \in \mathcal{U}_d$, populations of size d.
  - Swap random operator: **particular case** of deterministic swap intensity functions

$$L^{\mathrm{sw}} f(z) = \sum_{\substack{i,j=1 \\ i \neq j}}^{p} \big(f(z+\mathbf{e}_j-\mathbf{e}_i)-f(z)\big)\mu^{(i,j)}(\cancel{\alpha}, \cancel{s}, z), \quad \forall s \geqslant 0, \; z \in \mathbb{N}^p,$$

  - L^{sw} is the infinitesimal generator of a pure Markov swap process.
- **Interpretation of (5)**: $\Gamma_s([\bar{N}^{\mathrm{dem}}]_s, L^{\mathrm{sw}} f) = 0$: Γ_s is an invariant measure of the pure Markov swap.

Link with pure swap processes

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- **General case**: Γ_s is an invariant measure of a fictitious Markov pure swap of generator $\mathcal{L} = L_s^{\mathrm{sw}}(\omega)$. " Frozen" random environment ((ω, s) is fixed).

Averaging result (I)

Assumption

$\forall n \geqslant 0$, $t \geqslant 0$, there exists a unique $(\mathcal{G}_t)$-random probability kernel $(\pi_t(\omega, n, \mathrm{d}z))$ on $\mathcal{U}_n$ such that $\forall\ f : \mathcal{U}_n \mapsto \mathbb{R}$:

$$\pi_t(n, L_t^{\mathrm{sw}} f) = 0, \quad \mathrm{P} \otimes \mathrm{d}s \text{ a.s.} \tag{8}$$

Proposition (partial)

Under Assumption (8), the **aggregated processes** $Z^{\epsilon,\natural}$ *converge in distribution to a BD process* $\mathcal{X}$ *of intensity:*

$$\lambda^b(t, \mathcal{X}_t) = \int_{\mathcal{U}_{\mathcal{X}_t}} \mu^{b,\natural}(t, z)\pi_t(\mathcal{X}_t, \mathrm{d}z), \quad \lambda^d(t, \mathcal{X}_t) = \int_{\mathcal{U}_{\mathcal{X}_t}} \mu^{d,\natural}(t, z)\pi_t(\mathcal{X}_t, \mathrm{d}z).$$

Averaging result (I)

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Theorem (K., El Karoui)

Under Assumption 8, the demographic counting processes $(\mathbf{N}^{\mathrm{dem},\epsilon})$ *converge in distribution to the solution* $\mathcal{N} = (\mathcal{N}^b, \mathcal{N}^d)$ *of:*

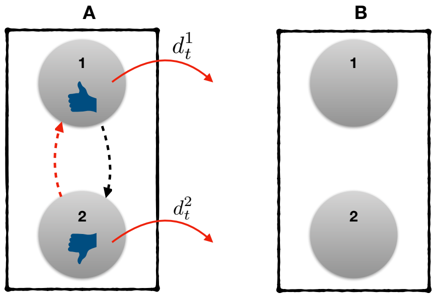
$$\mathrm{d}\mathcal{N}_t = \mathbf{Q}^{\mathrm{dem}}(\mathrm{d}t,]0, \pi_t(\mathcal{X}_{t^-}, \boldsymbol{\mu}^{\mathrm{dem}})]), \quad \forall t \geqslant 0, \tag{9}$$

with $\mathcal{X} = \sum_{i=1}^p N^{b,i} - \sum_{i=1}^p N^{d,i}$ *the limit of the aggregated process.*

Sketch of the proof

- It is sufficient to show that all stable limits $\bar{\mathbf{N}}^{\mathrm{dem}}$ have the same distribution.
- Assumption (8) + support property: $\bar{\mathbf{N}}^{\mathrm{dem}}$ has the $(\bar{G}_t)$ intensity $\pi_t(\mathcal{X}_{t^-}, \boldsymbol{\mu}^{\mathrm{dem}})$.
- Strong domination $\bar{\mathbf{N}}^{\mathrm{dem}} \prec \mathbf{G}^{\mathrm{dem}}$.
- Conclusion with "Converse result": *If X and X' are two counting processes strongly dominated by the same process Y and with same intensity functional, then X and X' have the same distribution.*

A toy example (I)

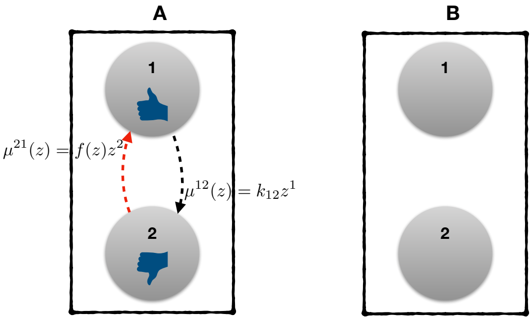


- Linear death functionals: $\mu^{d,i}(t, Z_t) = d_t^i Z_t^i$, $d_t^1 \leqslant d_t^2$

  (Aggregated death intensity) $\mu^{d,\natural}(t, Z_t) = d_t^1 Z_t^1 + d_t^2 Z_t^2$.

- If $Z_t^\natural = n$, individual death rate is $\dfrac{\mu^{d,\natural}(t, Z_t)}{n}$.

A toy example (II)



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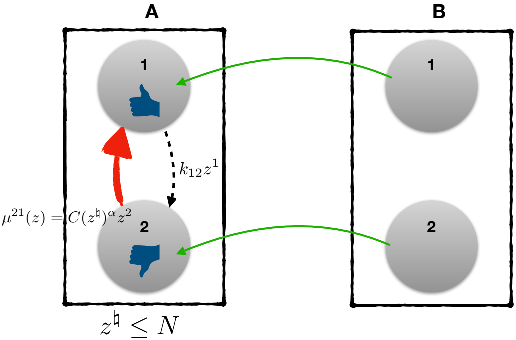
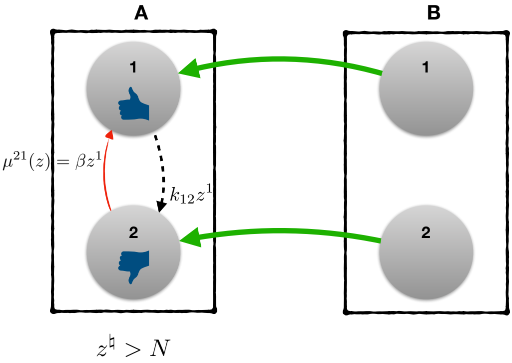


- Non-linear swap intensities.

$$\mu^{(1,2)}(\omega, t, z) = (k_t^{12}(\omega) z^\natural) z^1, \quad \mu^{(2,1)}(\omega, t, z) = k_t^{21}(\omega) z^2$$

A toy example (II)



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- Death intensity in the limit aggregated population:

$$\pi_t(n, \mu^{d,\natural}) = \left(d_t^1 p_t^1(n) + d_t^2 p_t^2(n)\right) n = \frac{d_t^1}{1 + \alpha_t n}(1 + \alpha_t w_t n) n,$$

with $w_t = \dfrac{d_t^2}{d_t^1}$.

Thank You For Your Attention