Recent developments in interest rate modelling

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Schedule of the course

- Friday 1 April 2022, 9.00 - 11.00, C. Fontana;
- Friday 8 April 2022, 11.15 - 12.15, C. Fontana;
- Friday 8 April 2022, 15.15 - 17.15, F. Mercurio (**on Zoom**);
- Friday 15 April 2022, 9.00 - 11.00, Z. Grbac.

Background: facts and figures

The interest rate market represents the largest portion of the OTC derivatives market: in the first half of 2021, the notional amount outstanding of interest rate contracts was 488.099 USD bn, with respect to 609.996 USD bn for all contracts.[1] 80% of the outstanding notional of OTC derivatives is on interest rates.

Over the last 10 years, several new phenomena appeared in interest rate markets:

- multi-curve environment;
- persistence of low (and even negative) rates;
- credit/liquidity risk in the interbank loans market and Libor manipulation;
- Libor reform and new alternative risk-free rates (SOFR, SONIA, €STR, etc.)

In this course, we aim at discussing how these phenomena have led and are leading to the development on new mathematical models.

[1] Source: BIS.

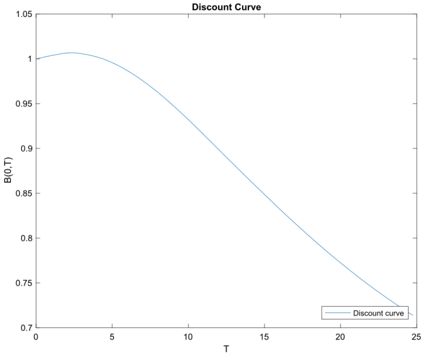
Outline

1. Basic notions of interest rates;
2. the multi-curve environment: stylized facts of post-crisis interest rate markets, terminology, basic traded assets;
3. absence of arbitrage in a multi-curve market;
4. a general multi-curve HJM framework;
5. models driven by affine processes and pricing aspects;
6. an overview of specific modelling approaches (short rate models, HJM models, market models, rational models);
7. the importance of stochastic discontinuities;
8. lecture by **Fabio Mercurio**: the Libor reform and its modelling aspects;
9. alternative risk-free rates and stochastic discontinuities;
10. an extended HJM framework for overnight and term rates;
11. an illustrative Vašiček example with stochastic discontinuities;
12. consistency and hedging issues in the presence of stochastic discontinuities.

Measuring the value of time

A fundamental purpose of interest rates is to **measure the value of time**:

- a **discount factor $P_t(T)$** measures the value at time t of one unit of currency delivered at time T, with $0 \leq t \leq T$, in the absence of any risk;
- since there is no risk, the terminal condition $P_T(T) = 1$ has to be satisfied;
- we associate $P_t(T)$ to the price of a **zero-coupon bond (ZCB)**;
- the **term structure** at time t is the collection $\{P_t(T); T \geq t\}$ and modelling the term structure involves describing its dynamics over time.



Term structure reconstructed on 25/06/2018, interpolated from OIS swaps.

Notions of interest rates

Starting from $\{P_t(T); T \geq t\}$, different types of interest rates can be defined:

- simple spot rate for $[S, T]$:

$$L(S, T) := \frac{1}{T - S}\left(\frac{1}{P_S(T)} - 1\right)$$

- simple forward rate for $[S, T]$, contracted at $t \leq S$:

$$L_t(S, T) := \frac{1}{T - S}\left(\frac{P_t(S)}{P_t(T)} - 1\right)$$

- continuously compounded forward rate for $[S, T]$, contracted at $t \leq S$:

$$F_t(S, T) := -\frac{\log P_t(T) - \log P_t(S)}{T - S}$$

- instantaneous forward rate with maturity T, contracted at $t \leq T$:

$$f_t(T) := -\frac{\partial}{\partial T} \log P_t(T)$$

- short rate at time t:

$$r_t := f_t(t)$$

References: Björk (2020), Musiela and Rutkowski (2005).

Classical modelling approaches

Depending on which notion of interest rate is taken as fundamental quantity, different modelling approaches arise:

1. **simple spot/forward rates** $\Rightarrow$ **Libor market models**:
   classically, the rate $L(S,T)$ was representing the Libor rate:
   - postulate dynamics for the process $(L_t(S,T))_{t\in[0,S]}$;
   - in the log-normal case, Black-type formulae for caps/floors;
   - calibration involves determining the volatility structure;
   - variant: forward price model, modelling directly $1+(T-S)L_t(S,T)$. This works especially well for low/negative interest rates, see Eberlein et al. (2020).
2. **instantaneous forward rates** $\Rightarrow$ **Heath-Jarrow-Morton (HJM) models**:
   arguably, the most general perspective on interest rate modelling:
   - postulate dynamics for $(f_t(T))_{t\in[0,T]}$, for all $T\in\mathbb{R}_+$;
   - this leads naturally to an infinite-dimensional system of SDEs...
   - ...or to a single SDE on a function space (Musiela parametrization);
   - HJM drift condition ensuring absence of arbitrage;
   - tractability: existence of finite-dimensional realizations (see Björk (2004)).

Classical modelling approaches

3. **short rate** $\Rightarrow$ **short rate models**:
   one of the most direct ways of modelling the term structure:
   - postulate dynamics for $(r_t)_{t\geq 0}$;
   - typically done directly under a risk-neutral measure Q;
   - compute ZCB prices and derivative prices by risk-neutral valuation:

   $$P_t(T) = E^Q\left[e^{-\int_t^T r_s \mathrm{d}s} \middle| \mathcal{F}_t\right]$$

   - often makes use of affine processes. Classical examples: Vasiček, Hull-White, Cox-Ingersoll-Ross, and many others, see e.g. Brigo and Mercurio (2006). Jiao et al. (2017) for persistently low interest rates, using α-stable processes.
4. **ZCB prices** $\Rightarrow$ **bond price models**:
   - postulate dynamics or a structural form for the term structure $\{P_t(T); T \geq t\}$;
   - Eberlein and Raible (1999) in the case of Lévy processes as drivers of $P_t(T)$;
   - potential models: Flesaker and Hughston (1996) and Rogers (1997), directly modeling the stochastic discount factor. This usually leads to rational models:

   $$P_t(T) = \frac{A(T) + B(T)X_t}{A(t) + B(t)X_t},$$

   where $(X_t)_{t\geq 0}$ is some Markovian factor process.

Libor rates after the global financial crisis

The London Interbank Offered Rate (Libor):

- daily computed as the trimmed average of rates reported by a panel of major banks for interbank loans, for five currencies (CHF, EUR, GBP, JPY, USD) and seven tenors (1D, 1W, 1M, 2M, 3M, 6M, 1Y);
- launched in 1986 and widely adopted as benchmark rate.

Prior to the 2007-2009 global financial crisis:

interbank loans among major banks $\approx$ *risk-free.*

Hence, the following two operations on $[S, T]$ should yield the same return:

1. interbank loan of 1 at S delivering $1 + (T - S)L(S, T)$ at T;
2. risk-free investment at S in $1/P_S(T)$ units of ZCB bonds with maturity T.

This implies the classical representation of Libor rates in terms of ZCB prices:

$$L(S, T) = \frac{1}{T - S}\left(\frac{1}{P_S(T)} - 1\right).$$

Post-crisis evidence:

$$L(S, T) \neq \frac{1}{T - S}\left(\frac{1}{P_S(T)} - 1\right).$$

Libor rates after the global financial crisis

Risks in the interbank market:

- counterparty risk;
- liquidity risk;
- funding and roll-over risk.

As a consequence, Libor rates cannot be considered representative of riskless loans.

The emergence of the **multiple curve environment**:

- Libor rates and risk-free ZCBs as distinct quantities;
- Libor rates used as benchmark rates to define derivatives' payoffs:
  $\Rightarrow$ one "curve" to represent Libor rates;
- risk-free ZCBs used as discount factors to compute (clean) derivatives prices:
  $\Rightarrow$ one "curve" to represent ZCB prices (or, equivalently, risk-free rates).

Assuming risk-neutral valuation, the price of an interest derivative is given by

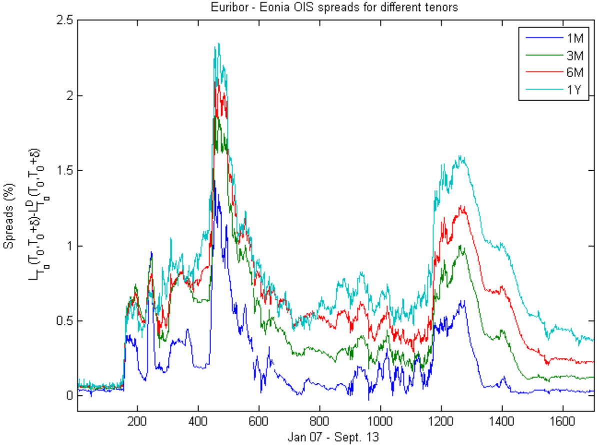
$$\Pi_t = P_t(T)\, E^{Q^T}\left[\Phi(L(S,T))\middle|\mathcal{F}_t\right],$$

where Φ represents a generic payoff function with maturity T and Q^T denotes the T-forward probability with numéraire $P(T)$.

Libor rates after the global financial crisis

Libor rates show a distinct behavior depending on the length of the loan (*tenor*): longer tenors are typically associated to greater risks.

Modelling consequence: one "curve" for each tenor $\delta \in \mathcal{D}$, where the set $\mathcal{D}$ of tenors is typically a subset of $\{1\mathrm{D}, 1\mathrm{W}, 1\mathrm{M}, 2\mathrm{M}, 3\mathrm{M}, 6\mathrm{M}, 1\mathrm{Y}\}$.



Differences (spreads) between Libor rates and simple spot OIS rates for different tenors.

The multi-curve market

To analyse a multi-curve market, we need to identify the **traded assets**:

- at least in theory, ZCBs can be considered as traded assets;
- however, in a multi-curve financial market, ZCBs do not suffice;
- Libor rates are benchmark rates and cannot be directly taken as traded assets;
- which contract can be considered as a basic traded asset related to Libor?

Forward rate agreement (FRA):
for $T \in \mathbb{R}_+$, $\delta \in \mathcal{D}$ and fixed rate $K \in \mathbb{R}$, the payoff at $T+\delta$ of a FRA is given by

$$\delta\big(L(T, T+\delta) - K\big).$$

The *forward Libor rate* $L_t(T, T+\delta)$ is the rate K such that the market value of the corresponding FRA at time t is null. The price of a generic FRA is then

$$\Pi_t^{\mathrm{FRA}}(T, \delta, K) = \delta P_t(T+\delta)\big(L_t(T, T+\delta) - K\big).$$

If we assume (but do not need to!) risk-neutral valuation, then

$$L_t(T, T+\delta) = E^{T+\delta}\big[L(T, T+\delta)\big|\mathcal{F}_t\big], \qquad \text{for } t \in [0, T].$$

References: Grbac and Runggaldier (2015), Cuchiero et al. (2016).

The multi-curve market

FRAs represent the **basic building block** for interest rate derivatives:

- linear derivatives (IRS, basis swaps) can be expressed in terms of FRAs;
- non-linear derivatives (caplets/floorlets, swaptions) can be considered as having FRAs as underlying assets.

We can then formalize the financial market as containing the following assets:

1. ZCBs for all maturities $T \in \mathbb{R}_+$;
2. FRAs for all maturities $T \in \mathbb{R}_+$, all tenors $\delta \in \mathcal{D}$, all rates $K \in \mathbb{R}$,

together with a numéraire asset with strictly positive price process $X^0 = (X_t^0)_{t \geq 0}$.

- This is a *Large Financial Market*, containing uncountably many assets;
- an appropriate notion of absence of arbitrage is *no asymptotic free lunch with vanishing risk* (**NAFLVR**), see Cuchiero et al. (2016).

Notation:

- $\mathcal{D}_0 := \mathcal{D} \cup \{0\}$;
- $\Pi_t^{\mathrm{FRA}}(T, 0, 0) := P_t(t \wedge T)$, for all $(t, T) \in \mathbb{R}_+^2$ and $K \in \mathbb{R}$.

The set of traded assets can then be indexed by $\mathcal{I}' := \mathbb{R}_+ \times \mathcal{D}_0 \times \mathbb{R}$.

NAFLVR in multi-curve markets

Since FRA prices are linear wrt. K, the set $\mathcal{I}'$ can be reduced to $\mathcal{I} := \mathbb{R}_+ \times \mathcal{D}_0$. In other words, it suffices to consider FRAs for an arbitrary *fixed* rate $\bar{K}$.

On a given stochastic basis $(\Omega, \mathcal{F}, \mathbb{F}, P)$, we proceed as follows:

- for all $n \in \mathbb{N}$, let $\boldsymbol{\mathcal{I}^n}$ be the family of all subsets $A \subseteq \mathcal{I}$ containing n elements;
- for each $A = ((T_1, \delta_1), \dots, (T_n, \delta_n)) \in \mathcal{I}^n$, let $\mathbf{S}^A = (S^1, \dots, S^n)$ be defined by
$$S^i_t = (X^0_t)^{-1} \Pi^{\mathrm{FRA}}_t(T_i, \delta_i, \bar{K}), \qquad \text{for } i = 1, \dots, n.$$
- assume that, for each $A \in \mathcal{I}^n$, $n \in \mathbb{N}$, the process $\mathbf{S}^A$ is a semimartingale;
- a predictable process $\boldsymbol{\theta} = (\theta^1, \dots, \theta^{|A|}) \in L_\infty(\mathbf{S}^A)$ is a 1-admissible trading strategy if $\boldsymbol{\theta}_0 = 0$ and $(\boldsymbol{\theta} \cdot \mathbf{S}^A)_t \geq -1$ a.s., for all $t \geq 0$;
- define
$$\mathcal{X}^A_1 := \left\{ \boldsymbol{\theta} \cdot \mathbf{S}^A : \boldsymbol{\theta} \in L_\infty(\mathbf{S}^A) \text{ and } \boldsymbol{\theta} \text{ is 1-admissible} \right\},$$
$$\mathcal{X}^n_1 := \bigcup_{A \in \mathcal{I}^n} \mathcal{X}^A_1 \qquad \text{and} \qquad \mathcal{X}_1 := \overline{\bigcup_{n \in \mathbb{N}} \mathcal{X}^n_1},$$
where the closure is taken in the Émery semimartingale topology;
- finally, the set of all **admissible portfolios** is given by
$$\mathcal{X} := \bigcup_{\lambda > 0} \lambda \mathcal{X}^1.$$

Reference: Fontana et al. (2020).

NAFLVR in multi-curve markets

Definition

The multi-curve financial market satisfies **NAFLVR** if

$$\overline{\boldsymbol{C}} \cap \boldsymbol{L}^{\infty}_{+} = \{\mathbf{0}\},$$

where $C := (K_0 - L^0_+) \cap L^\infty$, with $K_0 := \{X_\infty : X \in \mathcal{X}\}$ and $\overline{C}$ denoting the norm closure of C in L^∞.

Using the techniques of Cherny and Shiryaev (2005), we can obtain the following FTAP, extending the result of Cuchiero et al. (2016) to an infinite time horizon.

Theorem

The multi-curve financial market satisfies NAFLVR if and only if there exists an equivalent separating measure $\boldsymbol{Q}$, i.e., a probability measure $Q \sim P$ on $(\Omega, \mathcal{F})$ such that $\boldsymbol{E^Q[X_\infty] \leq 0}$ for all $X \in \mathcal{X}$.

Practical issue: characterizing an equivalent separating measure Q is difficult: a sufficient condition is $\exists$ of an equivalent local martingale measure (ELMM) for

$$(X^0)^{-1}\Pi^{\mathrm{FRA}}(T, \delta, \bar{K}), \qquad \text{for all } (T, \delta) \in \mathbb{R}_+ \times \mathcal{D}_0.$$

In concrete models, ELMMs can typically be explicitly characterized.

A weaker notion of no-arbitrage

Definition

The multi-curve financial market satisfies *no unbounded profit with bounded risk* (NUPBR) if the set $K_0^1 := \{X_\infty : X \in \mathcal{X}_1\}$ is bounded in probability.

- Introduced under this name in Karatzas and Kardaras (2007) and equivalent to some other notions of no-arbitrage (BK, NA1, see Kabanov et al. (2016));
- in large financial markets: Kardaras (2013) and Cuchiero et al. (2016);
- importance: minimal no-arbitrage condition for portfolio optimization.

Theorem

The multi-curve financial market satisfies NUPBR if and only if there exists an equivalent supermartingale deflator Z, i.e., a strictly positive supermartingale Z with $Z_0 = 1$ such that $Z(1+X)$ is a supermartingale for all $X \in \mathcal{X}_1$.

Remark: a sufficient condition for NUPBR is $\exists$ of an equivalent local martingale deflator (ELMD) Z, i.e., a strictly positive local martingale Z such that

$$Z\,(X^0)^{-1}\Pi^{\mathrm{FRA}}(T,\delta,\bar{K}) \in \mathcal{M}_{\mathrm{loc}}, \qquad \text{for all } (T,\delta) \in \mathbb{R}_+ \times \mathcal{D}_0.$$

In concrete models, usually the structure of Z can be explicitly described.
($\Rightarrow$ work in progress with E. Platen and S. Tappe.)

A general multi-curve HJM framework

Suppose that, on a given stochastic basis $(\Omega, \mathcal{F}, \mathbb{F}, P)$ we have

- a d-dimensional Brownian motion $W = (W_t)_{t \geq 0}$;
- an integer-valued random measure $\mu(\mathrm{d}t, \mathrm{d}x)$, with compensator $\nu(\mathrm{d}t, \mathrm{d}x) = \lambda_t(\mathrm{d}x)\mathrm{d}t$, where $\lambda_t(\mathrm{d}x)$ is a kernel from $(\Omega \times \mathbb{R}_+, \mathcal{P})$ into $(E, \mathcal{B}_E)$. We denote $\tilde{\mu}(\mathrm{d}t, \mathrm{d}x) := \mu(\mathrm{d}t, \mathrm{d}x) - \lambda_t(\mathrm{d}x)\mathrm{d}t$.

We assume the validity of the following **martingale representation** assumption.

Assumption

Every local martingale $N = (N_t)_{t \geq 0}$ can be represented as

$$N = N_0 + \theta \cdot W + \psi * \tilde{\mu},$$

for some $\theta \in L^2_{\mathrm{loc}}(W)$ and $\psi \in \mathcal{G}_{\mathrm{loc}}(\mu)$, see Jacod and Shiryaev (2003).

For simplicity, we assume that the **numéraire** is a savings account:

$$X^0 = \exp\left(\int_0^{\cdot} r_s \,\mathrm{d}s\right),$$

with $r = (r_t)_{t \geq 0}$ representing the risk-free short rate (typically, OIS rate).

Reference: Fontana et al. (2020).

An alternative representation of FRA prices

Let us recall the model-free representation of FRA prices:

$$\Pi_t^{\mathrm{FRA}}(T,\delta,K) = \delta P_t(T+\delta)\big(L_t(T,T+\delta) - K\big),$$

which we rewrite as follows, using the notation $K(\delta) := 1 + \delta K$:

$$\begin{aligned}\Pi_t^{\mathrm{FRA}}(T,\delta,K) &= P_t(T+\delta)\big(1+\delta L_t(T,T+\delta)\big) - K(\delta)P_t(T+\delta) \\ &= S_t^\delta P_t(T,\delta) - K(\delta)P_t(T+\delta),\end{aligned}$$

with

$$\boldsymbol{P_t(T,\delta)} := \frac{P_t(T+\delta)}{P_t(t+\delta)}\frac{1+\delta L_t(T,T+\delta)}{1+\delta L_t(t,t+\delta)}$$

and

$$\boldsymbol{S_t^\delta} := P_t(t+\delta)\big(1+\delta L_t(t,t+\delta)\big) =: \frac{1+\delta L_t(t,t+\delta)}{1+\delta L^{\mathrm{zcb}}(t,t+\delta)},$$

where L^{zcb} denotes the simple forward rate associated to risk-free ZCBs.

Terminology and interpretation:

1. S_t^δ: **spot multiplicative spread**, measures the relative riskiness of interbank rates with tenor δ at time t;
2. $P_t(T,\delta)$: δ-**tenor bond**, time-to-maturity behavior for tenor δ.

An alternative representation of FRA prices

These quantities admit a **foreign exchange analogy**:
let us imagine that a **foreign economy is associated to each tenor $\delta \in \mathcal{D}$**:

1. $P_t(T, \delta)$ represents the price of a ZCB of the foreign economy δ measured in units of the corresponding foreign currency;
2. S_t^δ represents the spot exchange rate between the foreign currency of economy δ and the domestic currency.

Then, the price of a foreign ZCB in units of the domestic currency is given by $S_t^\delta P_t(T, \delta)$ and the FRA becomes analogous to a FX forward contract.

Remark: this analogy suggests that this general HJM framework can be applied to other markets having multiple term structures, such as

- foreign exchange markets;
- energy markets;
- credit rating markets.

Remark: the classical single-curve setting corresponds to

$$S_t^\delta \equiv 1 \qquad \text{and} \qquad P_t(T, \delta) = P_t(T).$$

A general multi-curve HJM framework

We adopt the parametrization in terms of S_t^δ and $P_t(T,\delta)$ and suppose that

$$S_t^\delta = S_0^\delta\,\mathcal{E}\biggl(\int_0^\cdot \alpha_s^\delta\,\mathrm{d}s + \int_0^\cdot H_s^\delta \mathrm{d}W_s + \int_0^\cdot \int_E L^\delta(s,x)\tilde{\mu}(\mathrm{d}s,\mathrm{d}x)\biggr)$$

and, for all $\delta \in \mathcal{D}_0$ and $0 \leq t \leq T < +\infty$,

$$P_t(T,\delta) = \exp\biggl(-\int_t^T f_t(u,\delta)\,\mathrm{d}u\biggr),$$

where

$$\begin{aligned} f_t(T,\delta) = f_0(T,\delta) &+ \int_0^t a(s,T,\delta)\mathrm{d}s + \int_0^t b(s,T,\delta)\mathrm{d}W_s \\ &+ \int_0^t \int_E g(s,x,T,\delta)\tilde{\mu}(\mathrm{d}s,\mathrm{d}x). \end{aligned}$$

Technical assumptions: suitable integrability assumptions that ensure the applicability of ordinary and stochastic Fubini theorems to develop $\int_t^T f_t(u,\delta)\,\mathrm{d}u$.
(see Assumption 3.3 in Fontana et al. (2020) for details)

A general multi-curve HJM framework

Let us introduce the following notation, for all $0 \leq t \leq T$, $\delta \in \mathcal{D}_0$ and $x \in E$:

$$\bar{a}(t,T,\delta) := \int_t^T a(t,u,\delta)\mathrm{d}u, \;\; \bar{b}(t,T,\delta) := \int_t^T b(t,u,\delta)\mathrm{d}u, \;\; \bar{g}(t,x,T,\delta) := \int_t^T g(t,x,u,\delta)\mathrm{d}u.$$

Lemma

For every $T \in \mathbb{R}_+$ and $\delta \in \mathcal{D}_0$, it holds that

$$\begin{aligned} P(T,\delta) = P_0(T,\delta)\,\mathcal{E}\biggl(&\int_0^{\cdot} f_s(s,\delta)\mathrm{d}s - \int_0^{\cdot} \bar{a}(s,T,\delta)\mathrm{d}s + \frac{1}{2}\int_0^{\cdot} |\bar{b}(s,T,\delta)|^2\mathrm{d}s \\ &- \int_0^{\cdot} \bar{b}(s,T,\delta)\mathrm{d}W_s - \int_0^{\cdot}\int_E \bar{g}(s,x,T,\delta)\tilde{\mu}(\mathrm{d}s,\mathrm{d}x) \\ &+ \int_0^{\cdot}\int_E \bigl(e^{-\bar{g}(s,x,T,\delta)} - 1 + \bar{g}(s,x,T,\delta)\bigr)\mu(\mathrm{d}s,\mathrm{d}x)\biggr). \end{aligned}$$

By martingale representation, every density process $Z = (Z_t)_{t \geq 0}$ can be written as

$$Z = \mathcal{E}\bigl(-\theta \cdot W - \psi * \tilde{\mu}\bigr),$$

for some $\theta \in L^2_{\mathrm{loc}}(W)$ and $\psi : \Omega \times \mathbb{R}_+ \times E \to (-\infty, 1)$ belonging to $\mathcal{G}_{\mathrm{loc}}(\mu)$.

objective: characterize when Z is the density process of an ELMM Q.

A general multi-curve HJM framework

Let us define

$$\Lambda^*(t,x,T,\delta) := \big(1-\psi(t,x)\big)\big((1+L^\delta(t,x))e^{-\bar{g}(t,x,T,\delta)}-1\big) - L^\delta(t,x) + \bar{g}(t,x,T,\delta).$$

Proposition

Let $Q \sim P$ be a probability measure with density process Z represented as above. Then, Q is an ELMM if and only if, for all $T>0$,

$$\int_0^T \int_E |\Lambda^*(s,x,T,\delta)|\lambda_s(\mathrm{d}x)\mathrm{d}s < +\infty \text{ a.s.},$$

and the following two conditions hold a.s.

1. for a.e. $t \in \mathbb{R}_+$, it holds that

$$r_t = f_t(t,0),$$
$$\alpha_t^\delta = f_t(t,0) - f_t(t,\delta) + \theta_t^\top H_t^\delta + \int_E \psi(t,x)L^\delta(t,x)\lambda_t(\mathrm{d}x);$$

A general multi-curve HJM framework

Proposition (cont.)

2. for every $T > 0$ and for a.e. $t \in [0, T]$, it holds that

$$\bar{a}(t, T, \delta) = \frac{1}{2}|\bar{b}(t, T, \delta)|^2 + \bar{b}(t, T, \delta)^\top(\theta_t - H_t^\delta) + \int_E \Big((1 - \psi(t, x))(1 + L^\delta(t, x))(e^{-\bar{g}(t,x,T,\delta)} - 1) + \bar{g}(t, x, T, \delta)\Big)\lambda_t(\mathrm{d}x).$$

Proof (sketch):

- using the preceding Lemma and Yor's formula, write $Z(X^0)^{-1}S^\delta P(T, \delta)$ as a stochastic exponential $\mathcal{E}(Y)$, where the process Y can be explicitly computed;
- $\mathcal{E}(Y) \in \mathcal{M}_{\text{loc}} \iff Y \in \mathcal{M}_{\text{loc}}$;
- $Y \in \mathcal{M}_{\text{loc}}$ is equivalent to
  - Y has finite variation terms of locally integrable variation,
  - the predictable compensator Y^p of Y must be null;
- deduce that $Y^p \equiv 0 \iff$ HJM conditions (1)-(2).

Reference: follows from a more general result in Fontana et al. (2020).

A general multi-curve HJM framework

Interpretation:

1. condition (1) means the following:
   - the instantaneous yield on a ZCB must equal the risk-free short rate r_t;
   - the instantaneous yield on the floating leg of a FRA must equal the instantaneous risk-free return r_t plus a risk premium term.
2. condition (2) is a generalization of the HJM drift restriction.

Remark: conditions (1)-(2) actually characterize ELMDs, i.e., all strictly positive $Z \in \mathcal{M}_{\mathrm{loc}}$ such that

$$Z\,(X^0)^{-1}S^{\delta}P(T,\delta)$$

is a local martingale, for all $(T,\delta) \in \mathbb{R}_+ \times \mathcal{D}_0$, with $S^0 \equiv 1$ and $P(T,0) := P(T)$. Therefore, the Proposition can be used to deduce explicit conditions guaranteeing NUPBR for the multi-curve market.

A hybrid LMM-HJM framework

In the spirit of **Libor market models (LMM)**, let us denote for each $\delta \in \mathcal{D}$:

- $\mathcal{T}^\delta = \{T_0^\delta, \dots, T_{N^\delta}^\delta\}$ the set of settlement dates of traded FRAs with tenor δ;
- we assume that $T_0^\delta = T_0$ and $T_{N^\delta}^\delta = T^*$, for all $\delta \in \mathcal{D}$, for $T^* \in (0, +\infty)$;
- equidistant tenor structures: $T_i^\delta - T_{i-1}^\delta = \delta$, for all $i = 1, \dots, N^\delta$;
- $\mathcal{T} := \bigcup_{\delta \in \mathcal{D}} \mathcal{T}^\delta$, corresponding to the set of traded FRAs;
- ZCBs are traded for all maturities in the set $\mathcal{T}^0 := \mathcal{T} \cup \{T^* + \delta; \delta \in \mathcal{D}\}$.

Under the above structure, we are considering finitely many traded assets.

In the spirit of LMM, we postulate **dynamics directly for the forward Libor rates**, for every $\delta \in \mathcal{D}$ and $T \in \mathcal{T}^\delta$:

$$
\begin{aligned}
L_t(T, T+\delta) = L_0(T, T+\delta) &+ \int_0^t a^L(s, T, \delta)\mathrm{d}s + \int_0^t b^L(s, T, \delta)\mathrm{d}W_s \\
&+ \int_0^t \int_E g^L(s, x, T, \delta)\tilde{\mu}(\mathrm{d}s, \mathrm{d}x),
\end{aligned}
$$

for $b^L(\cdot, T, \delta) \in L^2_{\mathrm{loc}}(W)$ and $g^L(\cdot, \cdot, T, \delta) \in \mathcal{G}_{\mathrm{loc}}(\mu)$.

A hybrid LMM-HJM framework

Proposition

Suppose that the conditions of the previous Proposition are satisfied for $\delta = 0$ and for all $T \in \mathcal{T}^0$. Let Q be a probability measure with density process Z as represented above. Then, Q is an ELMM for all traded FRAs if and only if

$$\int_0^T \int_E \left| g^L(s,x,T,\delta)\left((1-\psi(s,x))e^{-\bar{g}(s,x,T+\delta,0)}-1\right)\right| \lambda_s(\mathrm{d}x)\mathrm{d}s < +\infty \text{ a.s.},$$

and the following condition holds a.s., for all $\delta \in \mathcal{D}$, $T \in \mathcal{T}^\delta$ and a.e. $t \in [0,T]$:

$$\begin{aligned} a^L(t,T,\delta) &= b^L(t,T,\delta)^\top \left(\theta_t + \bar{b}(t,T+\delta,0)\right) \\ &\quad - \int_E g^L(t,x,T,\delta)\left((1-\psi(t,x))e^{-\bar{g}(t,x,T+\delta,0)}-1\right)\lambda_t(\mathrm{d}x). \end{aligned}$$

Proof (sketch):

- the assumptions imply that $Z\,(X^0)^{-1}P(T+\delta)$ is a local martingale;
- apply the product rule to $L(T,T+\delta)Z\,(X^0)^{-1}P(T+\delta)$;
- apply similar reasoning as in the previous Proposition to characterize the local martingale property by analysing the finite variation terms.

Towards tractable models

So far, we discussed general dynamic multi-curve term structure models. We now move towards **tractable specifications** that allow for explicit **pricing** formulas.

Let us recall the concept of **spot multiplicative spread**:

$$S_t^\delta = \frac{1+\delta L_t(t,t+\delta)}{1+\delta L_t^{\mathrm{zcb}}(t,t+\delta)}.$$

Looking at market data, multiplicative spreads show a typical behavior:

- $S_t^{\delta_i} \geq 1$, for all $i = 1, \ldots, m$;
- $S_t^{\delta_i} \leq S_t^{\delta_j}$, for all $i,j = 1, \ldots, m$ such that $\delta_i < \delta_j$.

To develop a tractable class of models, we shall proceed as follows:

- martingale modelling: work directly under a risk-neutral probability Q;
- as fundamental modelling quantities, consider
  1. the instantaneous short-rate $\boldsymbol{r}$ defining the savings account numéraire X^0;
  2. spot multiplicative spreads $\boldsymbol{S^\delta}$, for $\delta \in \mathcal{D}$;
- model r and $\log S^\delta$ as affine functions of an affine process X.

References: Henrard (2014) for parametrizing multiple curves via multiplicative spreads, see also Cuchiero et al. (2016) and Grbac and Runggaldier (2015).

Forward multiplicative spreads

We also define forward multiplicative spreads:

$$S_t^\delta(T) = \frac{1 + \delta L_t(T, T+\delta)}{1 + \delta L_t^{\mathrm{zcb}}(T, T+\delta)},$$

where

- $L_t(T, T+\delta)$ is the forward Libor rate,
- $L_t^{\mathrm{zcb}}(T, T+\delta)$ is the simple forward rate associated to risk-free ZCBs.

Using the concept of δ-tenor bonds, forward multiplicative spreads correspond to

$$S_t^\delta(T) = S_t^\delta \frac{P_t(T, \delta)}{P_t(T)}.$$

Lemma

Suppose that $P(T)/X^0 \in \mathcal{M}(Q)$, for all $T \in \mathbb{R}_+$. The following are equivalent:

1. the X^0-discounted (T, δ)-FRA price belongs to $\mathcal{M}(Q)$,
2. $(X^0)^{-1} S^\delta P(T, \delta) \in \mathcal{M}(Q)$,
3. $S^\delta(T) \in \mathcal{M}(Q^T)$,
4. $L(T, T+\delta) \in \mathcal{M}(Q^{T+\delta})$,

where Q^T and $Q^{T+\delta}$ denote respectively the T-fwd and $(T+\delta)$-fwd measures.

Proof: easily follows from definition of multiplicative spread and Bayes' formula.

Martingale modelling

Working directly under a risk-neutral probability Q corresponds to the following:

Assumption (MM - martingale modelling)

The X^0-discounted prices of basic traded assets (ZCBs for all maturities $T \in \mathbb{R}_+$ and FRAs for all maturities $T \in \mathbb{R}_+$ and tenors $\delta \in \mathcal{D}$) are martingales under Q.

In more practical terms (and making use of the previous Lemma), this means that

$$P_t(T) = E^Q[e^{-\int_t^T r_s \mathrm{d}s} \big| \mathcal{F}_t],$$

$$S_t^\delta(T) = E^{Q^T}[S_T^\delta | \mathcal{F}_t].$$

- Under MM, this justifies the choice of r and S^δ as main modelling quantities.
- At this stage, tractability depends on a suitable specification of r and S^δ.

<u>Reference</u>: Cuchiero et al. (2019).

Basics of affine processes

- Let $(\Omega, \mathcal{F}, \mathbb{F}, Q)$ be a filtered probability space, with $\mathbb{T} < +\infty$ a time horizon;
- let D be a non-empty closed convex subset of a real vector space V;
- let $X = (X_t)_{0 \leq t \leq \mathbb{T}}$ be an adapted time-homogeneous and conservative Markov process taking values in D, starting at $X_0 = x \in D^o$;
- denote by $\{p_t : D \times \mathcal{B}_D \to [0,1]; t \in [0, \mathbb{T}]\}$ its transition kernels;
- let

$$\mathfrak{U}_T := \left\{\zeta \in V + \mathrm{i}V : \mathbb{E}[e^{\langle \zeta, X_t \rangle}] < +\infty, \text{ for all } t \in [0, T]\right\}$$

and

$$\mathfrak{D} := \{(t, \zeta) \in [0, \mathbb{T}] \times (V + \mathrm{i}V) : \zeta \in \mathfrak{U}_t\}.$$

Definition

The Markov process X is said to be **affine** if

1. it is stochastically continuous, i.e., the transition kernels satisfy $\lim_{s \to t} p_s(x, \cdot) = p_t(\cdot, x)$ weakly on D, for every $(t, x) \in [0, \mathbb{T}] \times D$;
2. there exist functions ϕ and ψ such that, for any $T \in [0, \mathbb{T}]$ and $u \in \mathfrak{U}_T$,

$$\boldsymbol{E^Q\left[e^{\langle u, X_T \rangle}\right] = e^{\phi(T,u) + \langle \psi(T,u), x \rangle}}.$$

References: Duffie et al. (2003), Keller-Ressel and Mayerhofer (2015). A generalization (affine semimartingales) has been more recently introduced in Keller-Ressel et al. (2019).

Basics of affine processes

The Markov property of X implies that ϕ and ψ satisfy the **semiflow relations**:

$$\begin{aligned}\phi(t+s,u) &= \phi(t,u) + \phi(s,\psi(t,u)),\\ \psi(t+s,u) &= \psi(s,\psi(t,u)),\end{aligned}$$

for all $t, s \in [0, \mathbb{T}]$ with $s + t \leq \mathbb{T}$.

The stochastic continuity of X implies its **regularity** and, therefore, the following derivatives exist and are continuous at $u = 0$:

$$F(u) := \frac{\partial \phi(t,u)}{\partial t}\bigg|_{t=0} \qquad \text{and} \qquad R(u) := \frac{\partial \psi(t,u)}{\partial t}\bigg|_{t=0}.$$

Therefore, we can differentiate wrt. s the semiflow relations and evaluate them at $s = 0$, thus obtaining the following system of **Riccati ODEs**:

$$\begin{aligned}\partial_t \phi(t,u) &= F(\psi(t,u)), \qquad & \phi(0,u) &= 0,\\ \partial_t \psi(t,u) &= R(\psi(t,u)), \qquad & \psi(0,u) &= u.\end{aligned}$$

The functions F and R completely characterize the law of X and are therefore called the **functional characteristics** of X. They have a Lévy-Khintchine form.

Basics of affine processes

Lemma

Let X be an affine process and $R := \langle \lambda, X \rangle$. Then $Y := (X, \int_0^{\cdot} R_s \, \mathrm{d}s)$ is an affine process taking values in $D \times \mathbb{R}$. Moreover, it holds that

$$\boldsymbol{E}^{\boldsymbol{Q}}\left[e^{\langle u, X_T \rangle + v \int_0^T R_s \, \mathrm{d}s}\right] = e^{\tilde{\phi}(T,u,v) + \langle \tilde{\psi}(T,u,v), x \rangle},$$

whenever the expectation is finite, with $\tilde{\phi}$ and $\tilde{\psi}$ satisfying the following ODEs:

$$\begin{aligned} \partial_t \tilde{\phi}(t,u,v) &= F(\tilde{\psi}(t,u,v)), & \phi(0,u,v) &= 0, \\ \partial_t \tilde{\psi}(t,u,v) &= R(\tilde{\psi}(t,u,v)) + v\lambda, & \psi(0,u,v) &= u. \end{aligned}$$

Remarks:

- this Lemma is a crucial result in the applications of affine processes in interest rate modelling, with R playing the role of a short-rate;
- more generally, an analogous statement holds true whenever $Y = (X, Z)$ is an affine stochastic volatility process, see Keller-Ressel (2011).

Affine multi-curve models

Definition

Let $\ell : [0, \mathbb{T}] \to \mathbb{R}$, $\lambda \in V$, $\mathbf{c} = \{c_\delta; \delta \in \mathcal{D}\}$ a family of functions $c_\delta : [0, \mathbb{T}] \to \mathbb{R}$ and $\gamma = \{\gamma_\delta; \delta \in \mathcal{D}\} \in V^{|\mathcal{D}|}$. The tuple $(X, \ell, \lambda, \mathbf{c}, \gamma)$ is an **affine short rate multi-curve model** if

$$r_t = \ell(t) + \langle \lambda, X_t \rangle, \qquad \text{for all } t \in [0, \mathbb{T}],$$

$$\log S_t^\delta = c_\delta(t) + \langle \gamma_\delta, X_t \rangle, \qquad \text{for all } t \in [0, \mathbb{T}] \text{ and } \delta \in \mathcal{D}.$$

Structure:

- classical short-rate approach for the risk-free rate r;
- multiplicative spreads as exponentially affine functions of X.

Special case: spreads can be modelled via an **instantaneous spread rate s^δ**:

$$\log S_t^\delta = \int_0^t s_u^\delta \, \mathrm{d}u = \int_0^t q_\delta(X_u) \, \mathrm{d}u, \qquad \text{for } \delta \in \mathcal{D},$$

where $q_\delta : D \to \mathbb{R}$ is an affine function, for each $\delta \in \mathcal{D}$. This modelling approach has some similarities with stochastic intensity models in credit risk, see Chapter 2 in Grbac and Runggaldier (2015).

Reference: Cuchiero et al. (2019).

Affine multi-curve models

The role of the functions ℓ and $\mathbf{c}$ consists in **fitting the initial term structures**:

- $\{P_0^M(T); T \in \mathbb{R}_+\}$: term structure of ZCB prices;
- $\{S_0^{\delta,M}(T); T \in \mathbb{R}_+, \delta \in \mathcal{D}\}$: term structure of forward multiplicative spreads.

Definition

An affine short rate multi-curve model $(X, \ell, \lambda, \mathbf{c}, \gamma)$ is said to achieve an exact fit to the initially observed term structures if

$$P_0(T) = P_0^M(T) \quad \text{and} \quad S_0^{\delta}(T) = S_0^{\delta,M}(T), \quad \text{for all } T \in [0, \mathbb{T}] \text{ and } \delta \in \mathcal{D}.$$

Interpretation: model quantities = market data at $t = 0$.

Proposition

An affine short rate multi-curve model $(X, \ell, \lambda, \mathbf{c}, \gamma)$ achieves an exact fit to the initially observed term structures if and only if

$$\begin{aligned} \ell(t) &= f_0^M(t) - f_0^0(t), \\ c_\delta(t) &= \log S_0^{\delta,M}(t) - \log S_0^{\delta,0}(t), \end{aligned} \qquad \text{for all } t \in [0, \mathbb{T}] \text{ and } \delta \in \mathcal{D},$$

where the superscript 0 denotes quantities computed from the model $(X, 0, \lambda, \mathbf{0}, \gamma)$.

Reference: Brigo and Mercurio (2001) (and Cuchiero et al. (2019) in this context).

Affine multi-curve models

Proposition

Let $(X, \ell, \lambda, \mathbf{c}, \gamma)$ be an affine short rate multi-curve model. Then, ZCB prices and forward multiplicative spreads are given by

$$P_t(T) = \exp\big(\mathcal{A}^0(t, T) + \langle \mathcal{B}^0(T-t), X_t\rangle\big),$$
$$S_t^\delta(T) = \exp\big(\mathcal{A}^\delta(t, T) + \langle \mathcal{B}^\delta(T-t), X_t\rangle\big),$$

for all $0 \leq t \leq T \leq \mathbb{T}$ and $\delta \in \mathcal{D}$, where

$$\mathcal{A}^0(t, T) = -\int_t^T \ell(u)\,\mathrm{d}u + \tilde{\phi}(T-t, 0, -\lambda),$$
$$\mathcal{B}^0(T-t) = \tilde{\psi}(T-t, 0, -\lambda),$$
$$\mathcal{A}^\delta(t, T) = c_\delta(T) + \tilde{\phi}(T-t, \gamma_\delta, -\lambda) - \tilde{\phi}(T-t, 0, -\lambda),$$
$$\mathcal{B}^\delta(T-t) = \tilde{\psi}(T-t, \gamma_\delta, \lambda) - \tilde{\psi}(T-t, 0, -\lambda).$$

Proof:

1. for ZCB prices: direct application of the affine transform formula;
2. for multiplicative spreads: application of the affine transform formula together with the martingale property of $S^\delta(T)$ under the T-fwd. measure Q^T.

Pricing applications: linear derivatives

All linear derivatives can be directly priced in terms of $P(T)$ and $S^{\delta}(T)$.

- **forward rate agreements (FRAs)**:

$$\Pi_t^{\mathrm{FRA}}(T,\delta,K) = P_t(T)S_t^{\delta}(T) - (1+\delta K)P_t(T+\delta)$$

- **interest rate swap (IRS)**, exchanging a stream of cashflows indexed to the Libor rate with tenor δ against a stream of cashflows with a fixed rate K at dates $T_1,\dots,T_N$, with $T_n - T_{n-1} = \delta$, for all $n = 1,\dots,N$:

$$\Pi_t^{\mathrm{IRS}}(T_1,T_N,K) = \sum_{n=1}^{N}\big(P_t(T_{n-1})S_t^{\delta}(T_{n-1}) - (1+\delta K)P_t(T_n)\big)$$

- **basis swap**, corresponding to a long/short position on two interest rate swaps with different tenors $\delta_1 < \delta_2$ and fixed leg with payment frequency δ_3:

$$\begin{aligned}\Pi_t^{\mathrm{BS}}(\mathcal{T}^1,\mathcal{T}^2,\mathcal{T}^3,K) = &\sum_{n=1}^{N_1}\big(P_t(T^1_{n-1})S_t^{\delta_1}(T^1_{n-1}) - P_t(T^1_n)\big) \\ &- \sum_{i=1}^{N_2}\big(P_t(T^2_{i-1})S_t^{\delta_2}(T^2_{i-1}) - P_t(T^2_i)\big) - \delta_3\,K\sum_{j=1}^{N_3}P_t(T^2_j),\end{aligned}$$

where $\mathcal{T}^i = \{T^i_0, T^i_1,\dots,T^1_{N_i}\}$, for $i = 1,2,3$, with $T^1_{N_1} = T^2_{N_2} = T^3_{N_3}$.

Remark: in pre-crisis setup (single-curve), value of a basis swap with $K = 0$ is null!

Reference: Grbac and Runggaldier (2015) and Appendix A of Cuchiero et al. (2019).

Pricing applications: non-linear derivatives

Non-linear derivatives can be priced by **Fourier methods**, see e.g. Chapter 10 in Filipović (2009). Let us consider the case of a **caplet** with payoff

$$\delta\big(L_T(T,T+\delta)-K\big)^+, \qquad \text{at maturity } T+\delta.$$

By risk-neutral valuation, the corresponding risk-neutral price is given by

$$\begin{aligned}\Pi_t^{\mathrm{CPL}}(T,\delta,K) &= \delta E\left[e^{-\int_t^{T+\delta} r_s \mathrm{d}s}\big(L_T(T,T+\delta)-K\big)^+\Big|\mathcal{F}_t\right]\\ &= P_t(T+\delta)E^{Q^{T+\delta}}\left[\big(e^{\mathcal{Y}_T}-(1+\delta K)\big)^+\big|\mathcal{F}_t\right],\end{aligned}$$

where

$$\begin{aligned}\mathcal{Y}_T &:= \log(S_T^\delta/P_T(T+\delta))\\ &= c_\delta(T)+\int_0^\delta \ell(T+u)\mathrm{d}u-\tilde{\phi}(T+\delta-t,0,-\lambda)+\langle\gamma_\delta-\tilde{\psi}(T+\delta-t,0,-\lambda),X_t\rangle.\end{aligned}$$

Let

$$\mathcal{C}_T := \left\{\nu\in\mathbb{R} : E^{T+\delta}\left[e^{\nu\mathcal{Y}_T}\right]<+\infty\right\}$$

and $\Lambda_T := \{\zeta\in\mathbb{C} : -\mathrm{Im}(\zeta)\in\mathcal{C}_T^o\}$. For $\zeta\in\Lambda_T$, we can compute the modified moment generating function of $\mathcal{Y}_T$:

$$\Phi_{\mathcal{Y}_T}(\zeta) := P_t(T+\delta)E^{Q^{T+\delta}}\left[e^{\mathrm{i}\zeta\mathcal{Y}_T}\big|\mathcal{F}_t\right],$$

with explicit representation as time-dependent exponentially affine function of X_t.

Pricing applications: non-linear derivatives

Proposition

Let $\zeta \in \mathbb{C}$, $\varepsilon \in \mathbb{R}$, $K(\delta) := 1 + \delta K$ and assume that $1+\varepsilon \in \mathcal{C}_T^o$. Then, the risk-neutral price of a caplet is given by

$$\Pi_t^{\mathrm{CPL}}(T,\delta,K) = \frac{1}{X_t^0}\left(\frac{1}{\pi}\int_{0-\mathrm{i}\varepsilon}^{+\infty-\mathrm{i}\varepsilon} \mathrm{Re}\left(e^{-\mathrm{i}\zeta \log K(\delta)} \frac{\Phi_{\mathcal{Y}_T}(\zeta-\mathrm{i})}{-\zeta(\zeta-\mathrm{i})}\right) \mathrm{d}\zeta + \mathcal{R}(\varepsilon)\right),$$

where $\mathcal{R}(\varepsilon)$ denotes a reminder term which depends on $K(\delta)$ and ε and satisfies $\mathcal{R}(\varepsilon) = 0$ for $\varepsilon > 0$.

Remarks:

- caplet pricing amounts to one-dimensional integration;
- computational effort can be further reduced by application of Fast Fourier Transform (FFT) methods, see Carr and Madan (1999);
- alternative methodology: Fourier-based quantization, Callegaro et al. (2019) (see also Fontana et al. (2021) for the specific application to caplets).

Reference: Cuchiero et al. (2019), relying on Theorem 5.1 of Lee (2004).

Pricing applications: non-linear derivatives

An alternative representation of a caplet price can be derived by a measure change. Let the probability $\widetilde{Q} \approx Q$ be defined by

$$\frac{\mathrm{d}\widetilde{Q}}{\mathrm{d}Q} := \frac{S_T^\delta}{X_T^0 S_0^\delta(T) P_0(T)} = \frac{S_T^\delta(T) P_T(T)}{X_T^0 S_0^\delta(T) P_0(T)}.$$

Since Q is a risk-neutral measure, $\widetilde{Q}$ intuitively corresponds to the measure having the floating leg of a FRA as numéraire. By changing the measure, we can write

$$\begin{aligned}\Pi_t^{\mathrm{CPL}}(T,\delta,K) &= P_t(T+\delta) E^{Q^{T+\delta}}\Big[\big(e^{\mathcal{Y}_T} - (1+\delta K)\big)^+ \Big| \mathcal{F}_t\Big] \\ &= S_t^\delta(T) P_t(T) \widetilde{Q}_t\big(\mathcal{Y}_T > \log(1+\delta K)\big) \\ &\quad - (1+\delta K) P_t(T+\delta) Q_t^{T+\delta}\big(\mathcal{Y}_T > \log(1+\delta K)\big).\end{aligned}$$

For **specific models**, these conditional probabilities can be explicitly computed:

- Gaussian (Hull-White type) models;
- Cox-Ingersoll-Ross models;
- Wishart models (see Cuchiero et al. (2019)).

Pricing applications: non-linear derivatives

Another important class of Libor derivatives are **swaptions**. Consider a swaption written on an IRS starting at $T_0 = T$ with payment dates $T_1, \dots, T_N$, with $T_n - T_{n-1} = \delta$, for $n = 1, \dots, N$. The corresponding risk-neutral price is given by

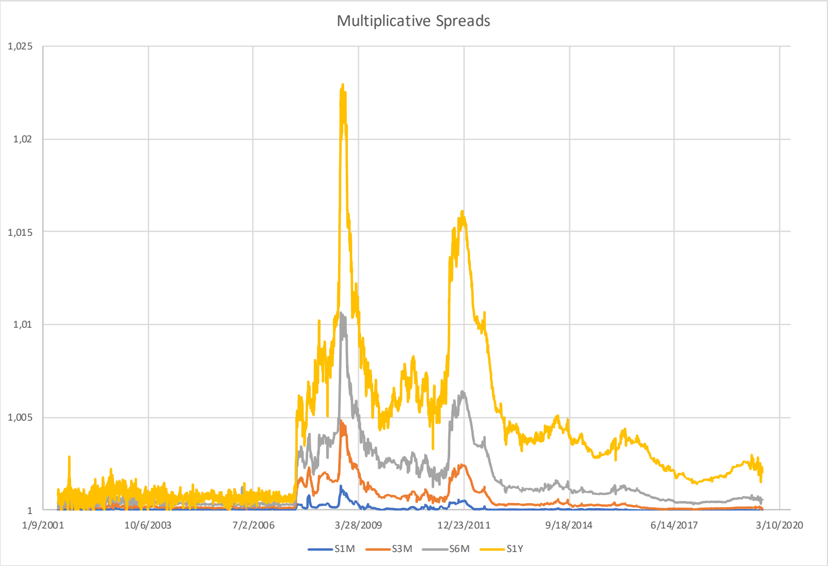
$$\Pi_t^{\mathrm{SWP}}(T_1, T_N, \delta, K) = E\left[e^{-\int_t^T r_s \mathrm{d}s} \left(\sum_{n=1}^{N} P_T(T_{n-1}) S_T^{\delta}(T_{n-1}) - (1 + \delta K) P_T(T_n) \right)^+ \middle| \mathcal{F}_t \right].$$

In **affine models**, the pricing of swaptions is challenging:

- approximate the exercise region, see Singleton and Umantsev (2002) and also Grbac et al. (2015) in the context of a multi-curve affine (Libor) model;
- lower bound that is quite close to the true value, see Caldana et al. (2017)

...otherwise: use a **polynomial process** as driver!

Looking back at multiplicative spreads



Looking back at multiplicative spreads

Empirical features of (multiplicative) spreads

- typically greater than one;
- longer tenors associated to larger spreads;
- volatility clustering and persistence of low values;
- strong comovements, in particular common upward jumps.

These phenomena can be reproduced in a model driven by **CBI processes**, which belong to the class of affine processes, see Duffie et al. (2003) and Li (2020).

Reference: Fontana et al. (2021).

A primer on CBI processes

Let $(\Omega, \mathcal{F}, \mathbb{F}, Q)$ be a filtered probability space supporting:

- a white noise $W(\mathrm{d}s, \mathrm{d}u)$ on $(0, +\infty)^2$ with intensity $\mathrm{d}s\,\mathrm{d}u$;
- a Poisson time-space random measure $M(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u)$ on $(0, +\infty)^3$ with intensity $\mathrm{d}s\,\pi(\mathrm{d}z)\,\mathrm{d}u$, let $\widetilde{M}(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u)$ be the compensated measure.

For each $i = 1, \ldots, m$, let $Y^i = (Y^i_t)_{t \geq 0}$ be the unique strong solution of

$$Y^i_t = y^i_0 + \int_0^t (\beta(i) - bY^i_s)\mathrm{d}s + \sigma \int_0^t \int_0^{Y^i_s} W(\mathrm{d}s, \mathrm{d}u) + \eta \int_0^t \int_0^{+\infty} \int_0^{Y^i_{s-}} z\widetilde{M}(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u),$$

where

- $\beta : \{1, \ldots, m\} \to \mathbb{R}_+$, with $\beta(i) \leq \beta(i+1)$;
- $(b, \sigma) \in \mathbb{R}^2$ and $\eta \geq 0$;
- π is a tempered alpha-stable measure:

$$\pi(\mathrm{d}z) = -\frac{1}{\Gamma(-\alpha)\cos(\alpha\pi/2)} \frac{e^{-\theta z}}{z^{1+\alpha}} \mathbf{1}_{\{z>0\}}\mathrm{d}z,$$

  with $\alpha \in (1, 2)$ and $\theta > \eta$.

Reference: Jiao et al. (2017) in the case of single-curve short rate modelling.

Modeling multiple curves via CBI processes

We specify the OIS short rate and spot multiplicative spreads by

$$r_t = \ell(t) + \mu^\top Y_t,$$
$$\log S_t^{\delta_i} = c_i(t) + Y_t^i,$$

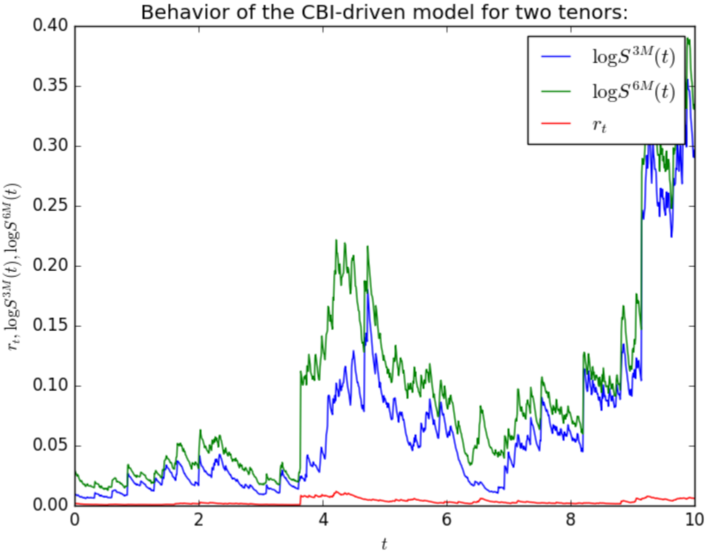
for all $t \geq 0$ and $i = 1, \ldots, m$, with $\ell : \mathbb{R}_+ \to \mathbb{R}$, $c_i : \mathbb{R}_+ \to \mathbb{R}_+$ and $\mu \in \mathbb{R}^m$.

- Functions ℓ and c_i are chosen to fit the term structures at $t = 0$;
- multiplicative spreads are by construction greater than one;
- OIS rate and spreads are driven by common sources of randomness;
- dependence among different spreads and OIS rates;
- each process Y^i is a self-exciting mean-reverting process;
- spreads have a mutually exciting: a large value of $S_t^{\delta_i}$ increases the likelihood of upward jumps of all spreads with tenor $\delta_j > \delta_i$.

Proposition

Suppose that $y_0^i \leq y_0^{i+1}$ and $c_i(t) \leq c_{i+1}(t)$, for all $i = 1, \ldots, m-1$ and $t \geq 0$. Then $S_t^{\delta_i}(T) \leq S_t^{\delta_{i+1}}(T)$ a.s., for all $i = 1, \ldots, m-1$ and $0 \leq t \leq T < +\infty$.

A sample path: multiplicative spreads



Affine structure of CBI-driven multi-curve models

CBI processes belong to the class of affine processes, see Duffie et al. (2003).

$$E\left[e^{-pY_t^i-q\int_0^t Y_s^i \mathrm{d}s}\right] = \exp\left(-Y_0^i v(t,p,q) - \beta(i)\int_0^t v(s,p,q)\,\mathrm{d}s\right),$$

where the function $v(\cdot,p)$ is given by the unique solution to the ODE

$$\partial_t v(t,p,q) = q - \phi\big(v(t,p,q)\big), \qquad v(0,p,q) = p,$$

with

$$\phi(z) = bz + \frac{\sigma^2}{2}z^2 + \frac{\theta^\alpha + z\alpha\eta\theta^{\alpha-1} - (z\eta+\theta)^\alpha}{\cos(\alpha\pi/2)}, \qquad \text{for } z \geq -\theta/\eta.$$

Theoretical results:

- existence of **exponential moments** of Y^i, in particular:

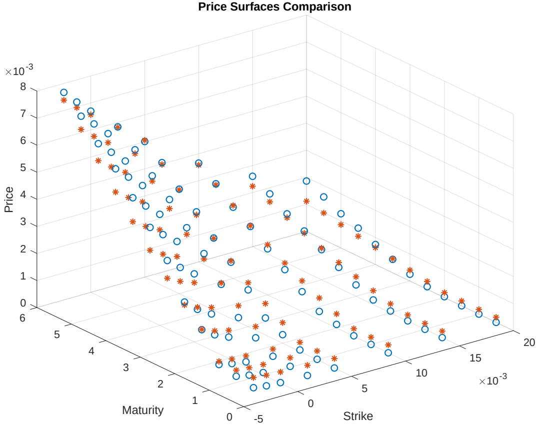
$$b \geq \frac{\sigma^2}{2}\frac{\theta}{\eta} + \eta\frac{(1-\alpha)\theta^{\alpha-1}}{\cos(\alpha\pi/2)} \qquad \Longrightarrow \qquad E[e^{Y_T^i}] < +\infty \quad \text{for all } T \geq 0.$$

- **0 is an inaccessible boundary** for Y^i if and only if $\beta(i) \geq \sigma^2/2$;
- characterization of the **ergodic distribution** of the process.

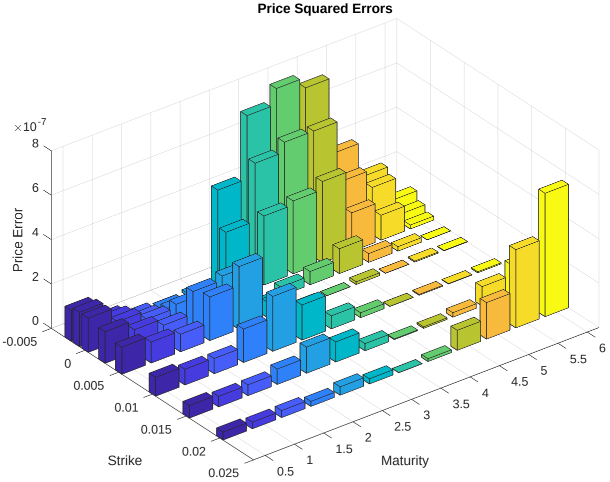
A calibration exercise

We calibrate a two-tenor (3M, 6M) version of the model. Data (25/06/2018):

- OIS and FRAs (bootstrapping vai Finmath Java library);
- market cap volatilities (Bachelier implied volatilities), maturities between 6 months and 6 years, strikes between -0.13% and 2%.



A calibration exercise



Björk, T. (2004), On the geometry of interest rate models, in Carmona, R. et al. (eds.), *Paris-Princeton Lectures on Mathematical Finance 2003*, pp. 133-215, Springer.

Björk, T. (2020), *Arbitrage Theory in Continuous Time*, 4th ed., Oxford University Press.

Bianchetti, M. and Morini, M. (eds.) (2013), *Interest Rate Modeling after the Financial Crisis*, Risk Books.

Brigo, D. and Mercurio, F. (2001), A deterministic-shift extension of analytically-tractable and time-homogeneous short-rate models, *Finance and Stochastics*, 5: 369–387.

Brigo, D. and Mercurio, F. (2006), *Interest Rate Models: Theory and Practice*, 2nd ed., Springer.

Callegaro, G., Fiorin, L. and Grasselli, M. (2019), Quantization meets Fourier: a new technology for pricing options, *Annals of Operations Research*, 282: 59–86.

Caldana, R., Fusai, G. and Gambaro, A. (2017), Approximate pricing of swaptions in affine and quadratic models, *Quantitative Finance*, 17(9): 1325–1345.

Carr, P. and Madan, D. (1999), Option valuation using the fast Fourier transform, *Journal of Computational Finance*, 2(4): 61–73.

Cherny, A. and Shiryaev, A. (2005), On stochastic integrals up to infinity and predictable criteria for integrability, in Émery, M. et al. (eds.), *Séminaire de Probabilités XXXVIII*, pp. 165–185, Springer.

Cuchiero, C., Fontana, C. and Gnoatto, A. (2016), A general HJM framework for multiple yield curve modeling, *Finance and Stochastics*, 20(2): 267–320.

Cuchiero, C., Fontana, C. and Gnoatto, A. (2019), Affine multiple yield curve models, *Mathematical Finance*, 29(2): 568–611.

Cuchiero, C., Klein, I. and Teichmann, J. (2016), A new perspective on the fundamental theorem of asset pricing for large financial markets, *Theory of Probability and its Applications*, 60: 561–579.

Duffie, D., Filipović, D. and Schachermayer, W. (2003), Affine processes and applications in finance, *Annals of Applied Probability*, 13(3): 984–1053.

Eberlein, E., Gerhart, C. and Grbac, Z. (2020), Multiple curve Lévy forward price model allowing for negative interest rates, *Mathematical Finance*, 30(1): 167–195.

Eberlein, E. and Raible, S. (1999), Term structure models driven by general Lévy processes, *Mathematical Finance*, 9(1): 31–53.

Filipović, D. (2009), *Term-Structure Models: A Graduate Course*, Springer.

Flesaker, B. and Hughston, L. (1996), Positive interest, *Risk Magazine*.

Fontana, C., Gnoatto, A. and Szulda, G. (2021), Multiple yield curve modelling with CBI processes, *Mathematics and Financial Economics*, 15: 579–610.

Fontana, C., Grbac, Z., Gümbel, S. and Schmidt, T. (2020), Term structure modeling for multiple curves with stochastic discontinuities, *Finance and Stochastics*, 24: 465–511.

Grbac, Z., Papapantoleon, A., Schoenmakers, J. and Skovmand, D. (2020), Affine LIBOR models with multiple curves: theory, examples and calibration, *SIAM Journal on Financial Mathematics*, 6: 984–1025.

Grbac, Z. and Runggaldier, W. (2015), *Interest Rate Modeling: Post-Crisis Challenges and Approaches*, Springer.

Henrard, M. (2014), *Interest Rate Modelling in the Multi-curve Framework*, Palgrave Macmillan.

Jacod, J. and Shiryaev, A. (2003), *Limit Theorems for Stochastic Processes*, 2nd ed., Springer.

Jiao, Y., Ma, C. and Scotti, S. (2017(, Alpha-CIR model with branching processes in sovereign interest rate modeling, *Finance and Stochastics*, 21: 789–813.

Kabanov, Y., Kardaras, C. and Song, S. (2016), No arbitrage of the first kind and local martingale numéraires, *Finance and Stochastics*, 20: 1097–1108.

Karatzas, I. and Kardaras, C. (2007), The numéraire portfolio in semimartingale financial models, *Finance and Stochastics*, 11: 447–493.

Kardaras, C. (2013), On the closure in the Émery topology of semimartingale wealth-process sets, *Annals of Applied Probability*, 23(4): 1355–1376.

Keller-Ressel, M. (2011), Moment explosion and long-term behavior of affine stochastic volatility models, *Mathematical Finance*, 21(1): 73–98.

Keller-Ressel, M. and Mayerhofer, E. (2015), Exponential moments of affine processes, *Annals of Applied Probability*, 25(2): 714–752.

Keller-Ressel, M., Schmidt, T. and Wardenga, R. (2019), Affine processes beyond stochastic continuity, *Annals of Applied Probability*, 29(6): 3387–3437.

Lee, R. (2004), Option pricing by transform methods: extensions, unification and error control, *Journal of Computational Finance*, 7(3): 51–86.

Li, Z. (2020), Continuous-state branching processes with immigration, in Jiao, Y. (ed.), *From Probability to Finance. Lecture Notes of BICMR Summer School on Financial Mathematics*. Springer.

Musiela, M. and Rutkowski, M. (2005), *Martingale Methods in Financial Modelling*, 2nd ed., Springer.

Rogers, L.C.G. (1997), The potential approach to the term structure of interest rates and foreign exchanges, *Mathematical Finance*, 7(2): 157–176.

Singleton, K.J. and Umantsev, L. (2002), Pricing coupon-bond options and swaptions in affine term structure models, *Mathematical Finance*, 12(4): 427–446.