Recent developments in interest rate modelling

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Schedule of the course

- Friday 1 April 2022, 9.00 - 11.00, C. Fontana;
- Friday 8 April 2022, 11.15 - 12.15, C. Fontana and Z. Grbac;
- Friday 8 April 2022, 15.15 - 17.15, F. Mercurio (**on Zoom**);
- Friday 15 April 2022, 9.00 - 11.00, C. Fontana and Z. Grbac.

Background: facts and figures

The interest rate market represents the largest portion of the OTC derivatives market: in the first half of 2021, the notional amount outstanding of interest rate contracts was 488.099 USD bn, with respect to 609.996 USD bn for all contracts.[1] 80% of the outstanding notional of OTC derivatives is on interest rates.

Over the last 10 years, several new phenomena appeared in interest rate markets:

- multi-curve environment;
- persistence of low (and even negative) rates;
- credit/liquidity risk in the interbank loans market and Libor manipulation;
- Libor reform and new alternative risk-free rates (SOFR, SONIA, €STR, etc.)

In this course, we aim at discussing how these phenomena have led and are leading to the development on new mathematical models.

[1] Source: BIS.

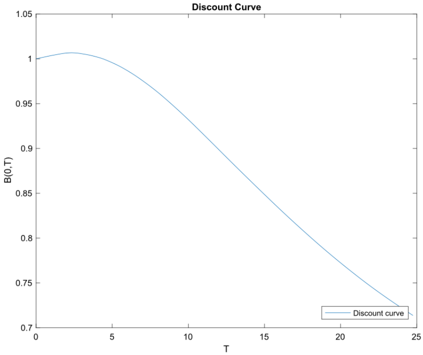
Outline

1. Basic notions of interest rates;
2. the multi-curve environment: stylized facts of post-crisis interest rate markets, terminology, basic traded assets;
3. absence of arbitrage in a multi-curve market;
4. a general multi-curve HJM framework;
5. models driven by affine processes and pricing aspects;
6. an overview of specific modelling approaches
(short rate models, HJM models, market models, rational models);
7. the importance of stochastic discontinuities;
8. lecture by **Fabio Mercurio**: the Libor reform and its modelling aspects;
9. alternative risk-free rates and stochastic discontinuities;
10. an extended HJM framework for overnight and term rates;
11. an illustrative Vasiček example with stochastic discontinuities;
12. consistency and hedging issues in the presence of stochastic discontinuities.

Measuring the value of time

A fundamental purpose of interest rates is to **measure the value of time**:

- a **discount factor $P_t(T)$** measures the value at time t of one unit of currency delivered at time T, with $0 \leq t \leq T$, in the absence of any risk;
- since there is no risk, the terminal condition $P_T(T) = 1$ has to be satisfied;
- we associate $P_t(T)$ to the price of a **zero-coupon bond (ZCB)**;
- the **term structure** at time t is the collection $\{P_t(T); T \geq t\}$ and modelling the term structure involves describing its dynamics over time.



Term structure reconstructed on 25/06/2018, interpolated from OIS swaps.

Notions of interest rates

Starting from $\{P_t(T); T \geq t\}$, different types of interest rates can be defined:

- simple spot rate for $[S, T]$:

$$L(S,T) := \frac{1}{T-S}\left(\frac{1}{P_S(T)} - 1\right)$$

- simple forward rate for $[S, T]$, contracted at $t \leq S$:

$$L_t(S,T) := \frac{1}{T-S}\left(\frac{P_t(S)}{P_t(T)} - 1\right)$$

- continuously compounded forward rate for $[S, T]$, contracted at $t \leq S$:

$$F_t(S,T) := -\frac{\log P_t(T) - \log P_t(S)}{T-S}$$

- instantaneous forward rate with maturity T, contracted at $t \leq T$:

$$f_t(T) := -\frac{\partial}{\partial T}\log P_t(T)$$

- short rate at time t:

$$r_t := f_t(t)$$

References: Björk (2020), Musiela and Rutkowski (2005).

Classical modelling approaches

Depending on which notion of interest rate is taken as fundamental quantity, different modelling approaches arise:

1. **simple spot/forward rates** $\Rightarrow$ **Libor market models**:
   classically, the rate $L(S,T)$ was representing the Libor rate:
   - postulate dynamics for the process $(L_t(S,T))_{t\in[0,S]}$;
   - in the log-normal case, Black-type formulae for caps/floors;
   - calibration involves determining the volatility structure;
   - variant: forward price model, modelling directly $1+(T-S)L_t(S,T)$. This works especially well for low/negative interest rates, see Eberlein et al. (2020).
2. **instantaneous forward rates** $\Rightarrow$ **Heath-Jarrow-Morton (HJM) models**:
   arguably, the most general perspective on interest rate modelling:
   - postulate dynamics for $(f_t(T))_{t\in[0,T]}$, for all $T\in\mathbb{R}_+$;
   - this leads naturally to an infinite-dimensional system of SDEs...
   - ...or to a single SDE on a function space (Musiela parametrization);
   - HJM drift condition ensuring absence of arbitrage;
   - tractability: existence of finite-dimensional realizations (see Björk (2004)).

Classical modelling approaches

3. **short rate** $\Rightarrow$ **short rate models**:
   one of the most direct ways of modelling the term structure:
   - postulate dynamics for $(r_t)_{t\geq 0}$;
   - typically done directly under a risk-neutral measure Q;
   - compute ZCB prices and derivative prices by risk-neutral valuation:
   $$P_t(T) = E^Q\left[e^{-\int_t^T r_s \mathrm{d}s} \middle| \mathcal{F}_t\right]$$
   - often makes use of affine processes. Classical examples: Vasiček, Hull-White, Cox-Ingersoll-Ross, and many others, see e.g. Brigo and Mercurio (2006). Jiao et al. (2017) for persistently low interest rates, using α-stable processes.
4. **ZCB prices** $\Rightarrow$ **bond price models**:
   - postulate dynamics or a structural form for the term structure $\{P_t(T);\, T \geq t\}$;
   - Eberlein and Raible (1999) in the case of Lévy processes as drivers of $P_t(T)$;
   - potential models: Flesaker and Hughston (1996) and Rogers (1997), directly modeling the stochastic discount factor. This usually leads to rational models:
   $$P_t(T) = \frac{A(T) + B(T)X_t}{A(t) + B(t)X_t},$$
   where $(X_t)_{t\geq 0}$ is some Markovian factor process.

Libor rates after the global financial crisis

The London Interbank Offered Rate (Libor):

- daily computed as the trimmed average of rates reported by a panel of major banks for interbank loans, for five currencies (CHF, EUR, GBP, JPY, USD) and seven tenors (1D, 1W, 1M, 2M, 3M, 6M, 1Y);
- launched in 1986 and widely adopted as benchmark rate.

Prior to the 2007-2009 global financial crisis:

$$\textit{interbank loans among major banks} \quad \approx \quad \textit{risk-free.}$$

Hence, the following two operations on $[S, T]$ should yield the same return:

1. interbank loan of 1 at S delivering $1 + (T - S)L(S, T)$ at T;
2. risk-free investment at S in $1/P_S(T)$ units of ZCB bonds with maturity T.

This implies the classical representation of Libor rates in terms of ZCB prices:

$$L(S, T) = \frac{1}{T - S}\left(\frac{1}{P_S(T)} - 1\right).$$

Post-crisis evidence:

$$L(S, T) \neq \frac{1}{T - S}\left(\frac{1}{P_S(T)} - 1\right).$$

Libor rates after the global financial crisis

Risks in the interbank market:

- counterparty risk;
- liquidity risk;
- funding and roll-over risk.

As a consequence, Libor rates cannot be considered representative of riskless loans.

The emergence of the **multiple curve environment**:

- Libor rates and risk-free ZCBs as distinct quantities;
- Libor rates used as benchmark rates to define derivatives' payoffs:
  $\Rightarrow$ one "curve" to represent Libor rates;
- risk-free ZCBs used as discount factors to compute (clean) derivatives prices:
  $\Rightarrow$ one "curve" to represent ZCB prices (or, equivalently, risk-free rates).

Assuming risk-neutral valuation, the price of an interest derivative is given by

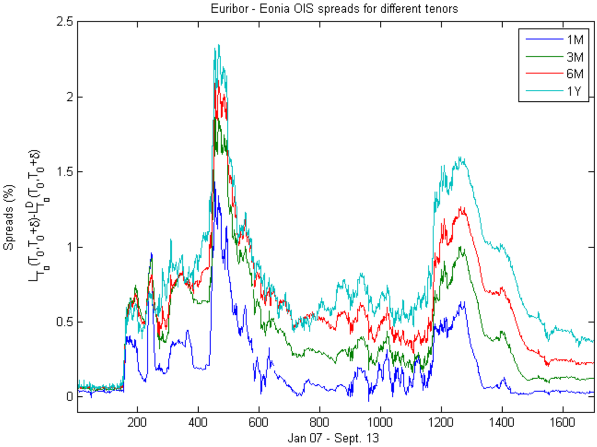
$$\Pi_t = P_t(T)\, E^{Q^T}\left[\Phi(L(S,T))\big|\mathcal{F}_t\right],$$

where Φ represents a generic payoff function with maturity T and Q^T denotes the T-forward probability with numéraire $P(T)$.

Libor rates after the global financial crisis

Libor rates show a distinct behavior depending on the length of the loan (*tenor*): longer tenors are typically associated to greater risks.

Modelling consequence: one "curve" for each tenor $\delta \in \mathcal{D}$, where the set $\mathcal{D}$ of tenors is typically a subset of $\{1\mathrm{D}, 1\mathrm{W}, 1\mathrm{M}, 2\mathrm{M}, 3\mathrm{M}, 6\mathrm{M}, 1\mathrm{Y}\}$.



Differences (spreads) between Libor rates and simple spot OIS rates for different tenors.

The multi-curve market

To analyse a multi-curve market, we need to identify the **traded assets**:

- at least in theory, ZCBs can be considered as traded assets;
- however, in a multi-curve financial market, ZCBs do not suffice;
- Libor rates are benchmark rates and cannot be directly taken as traded assets;
- which contract can be considered as a basic traded asset related to Libor?

Forward rate agreement (FRA):
for $T \in \mathbb{R}_+$, $\delta \in \mathcal{D}$ and fixed rate $K \in \mathbb{R}$, the payoff at $T+\delta$ of a FRA is given by

$$\delta\big(L(T, T+\delta) - K\big).$$

The *forward Libor rate* $L_t(T, T+\delta)$ is the rate K such that the market value of the corresponding FRA at time t is null. The price of a generic FRA is then

$$\Pi_t^{\mathrm{FRA}}(T, \delta, K) = \delta P_t(T+\delta)\big(L_t(T, T+\delta) - K\big).$$

If we assume (but do not need to!) risk-neutral valuation, then

$$L_t(T, T+\delta) = E^{T+\delta}\big[L(T, T+\delta)\big|\mathcal{F}_t\big], \qquad \text{for } t \in [0, T].$$

References: Grbac and Runggaldier (2015), Cuchiero et al. (2016).

The multi-curve market

FRAs represent the **basic building block** for interest rate derivatives:

- linear derivatives (IRS, basis swaps) can be expressed in terms of FRAs;
- non-linear derivatives (caplets/floorlets, swaptions) can be considered as having FRAs as underlying assets.

We can then formalize the financial market as containing the following assets:

1. ZCBs for all maturities $T \in \mathbb{R}_+$;
2. FRAs for all maturities $T \in \mathbb{R}_+$, all tenors $\delta \in \mathcal{D}$, all rates $K \in \mathbb{R}$,

together with a numéraire asset with strictly positive price process $X^0 = (X^0_t)_{t \geq 0}$.

- This is a *Large Financial Market*, containing uncountably many assets;
- an appropriate notion of absence of arbitrage is *no asymptotic free lunch with vanishing risk* (**NAFLVR**), see Cuchiero et al. (2016).

Notation:

- $\mathcal{D}_0 := \mathcal{D} \cup \{0\}$;
- $\Pi^{\mathrm{FRA}}_t(T, 0, 0) := P_t(t \wedge T)$, for all $(t, T) \in \mathbb{R}^2_+$ and $K \in \mathbb{R}$.

The set of traded assets can then be indexed by $\mathcal{I}' := \mathbb{R}_+ \times \mathcal{D}_0 \times \mathbb{R}$.

NAFLVR in multi-curve markets

Since FRA prices are linear wrt. K, the set $\mathcal{I}'$ can be reduced to $\mathcal{I} := \mathbb{R}_+ \times \mathcal{D}_0$. In other words, it suffices to consider FRAs for an arbitrary *fixed* rate $\bar{K}$.

On a given stochastic basis $(\Omega, \mathcal{F}, \mathbb{F}, P)$, we proceed as follows:

- for all $n \in \mathbb{N}$, let $\boldsymbol{\mathcal{I}^n}$ be the family of all subsets $A \subseteq \mathcal{I}$ containing n elements;
- for each $A = ((T_1, \delta_1), \ldots, (T_n, \delta_n)) \in \mathcal{I}^n$, let $\mathbf{S}^A = (S^1, \ldots, S^n)$ be defined by
$$S^i_t = (X^0_t)^{-1} \Pi^{\mathrm{FRA}}_t(T_i, \delta_i, \bar{K}), \qquad \text{for } i = 1, \ldots, n.$$
- assume that, for each $A \in \mathcal{I}^n$, $n \in \mathbb{N}$, the process $\mathbf{S}^A$ is a semimartingale;
- a predictable process $\boldsymbol{\theta} = (\theta^1, \ldots, \theta^{|A|}) \in L_\infty(\mathbf{S}^A)$ is a 1-admissible trading strategy if $\boldsymbol{\theta}_0 = 0$ and $(\boldsymbol{\theta} \cdot \mathbf{S}^A)_t \geq -1$ a.s., for all $t \geq 0$;
- define
$$\mathcal{X}^A_1 := \left\{ \boldsymbol{\theta} \cdot \mathbf{S}^A : \boldsymbol{\theta} \in L_\infty(\mathbf{S}^A) \text{ and } \boldsymbol{\theta} \text{ is 1-admissible} \right\},$$
$$\mathcal{X}^n_1 := \bigcup_{A \in \mathcal{I}^n} \mathcal{X}^A_1 \qquad \text{and} \qquad \mathcal{X}_1 := \overline{\bigcup_{n \in \mathbb{N}} \mathcal{X}^n_1},$$
where the closure is taken in the Émery semimartingale topology;
- finally, the set of all **admissible portfolios** is given by
$$\mathcal{X} := \bigcup_{\lambda > 0} \lambda \mathcal{X}^1.$$

Reference: Fontana et al. (2020).

NAFLVR in multi-curve markets

Definition

The multi-curve financial market satisfies **NAFLVR** if

$$\overline{C} \cap L^\infty_+ = \{0\},$$

where $C := (K_0 - L^0_+) \cap L^\infty$, with $K_0 := \{X_\infty : X \in \mathcal{X}\}$ and $\overline{C}$ denoting the norm closure of C in L^∞.

Using the techniques of Cherny and Shiryaev (2005), we can obtain the following FTAP, extending the result of Cuchiero et al. (2016) to an infinite time horizon.

Theorem

The multi-curve financial market satisfies NAFLVR if and only if there exists an equivalent separating measure Q, i.e., a probability measure $Q \sim P$ on $(\Omega, \mathcal{F})$ such that $E^Q[X_\infty] \leq 0$ for all $X \in \mathcal{X}$.

Practical issue: characterizing an equivalent separating measure Q is difficult: a sufficient condition is $\exists$ of an equivalent local martingale measure (ELMM) for

$$(X^0)^{-1}\Pi^{\mathrm{FRA}}(T, \delta, \bar{K}), \qquad \text{for all } (T, \delta) \in \mathbb{R}_+ \times \mathcal{D}_0.$$

In concrete models, ELMMs can typically be explicitly characterized.

A weaker notion of no-arbitrage

Definition

The multi-curve financial market satisfies *no unbounded profit with bounded risk* (NUPBR) if the set $K_0^1 := \{X_\infty : X \in \mathcal{X}_1\}$ is bounded in probability.

- Introduced under this name in Karatzas and Kardaras (2007) and equivalent to some other notions of no-arbitrage (BK, NA1, see Kabanov et al. (2016));
- in large financial markets: Kardaras (2013) and Cuchiero et al. (2016);
- importance: minimal no-arbitrage condition for portfolio optimization.

Theorem

The multi-curve financial market satisfies NUPBR if and only if there exists an equivalent supermartingale deflator Z, i.e., a strictly positive supermartingale Z with $Z_0 = 1$ such that $Z(1+X)$ is a supermartingale for all $X \in \mathcal{X}_1$.

Remark: a sufficient condition for NUPBR is $\exists$ of an equivalent local martingale deflator (ELMD) Z, i.e., a strictly positive local martingale Z such that

$$Z\,(X^0)^{-1}\Pi^{\mathrm{FRA}}(T,\delta,\bar{K}) \in \mathcal{M}_{\mathrm{loc}}, \qquad \text{for all } (T,\delta) \in \mathbb{R}_+ \times \mathcal{D}_0.$$

In concrete models, usually the structure of Z can be explicitly described.
($\Rightarrow$ work in progress with E. Platen and S. Tappe.)

A general multi-curve HJM framework

Suppose that, on a given stochastic basis $(\Omega, \mathcal{F}, \mathbb{F}, P)$ we have

- a d-dimensional Brownian motion $W = (W_t)_{t\geq 0}$;
- an integer-valued random measure $\mu(\mathrm{d}t, \mathrm{d}x)$, with compensator $\nu(\mathrm{d}t, \mathrm{d}x) = \lambda_t(\mathrm{d}x)\mathrm{d}t$, where $\lambda_t(\mathrm{d}x)$ is a kernel from $(\Omega \times \mathbb{R}_+, \mathcal{P})$ into $(E, \mathcal{B}_E)$. We denote $\tilde{\mu}(\mathrm{d}t, \mathrm{d}x) := \mu(\mathrm{d}t, \mathrm{d}x) - \lambda_t(\mathrm{d}x)\mathrm{d}t$.

We assume the validity of the following **martingale representation** assumption.

Assumption

Every local martingale $N = (N_t)_{t\geq 0}$ can be represented as

$$N = N_0 + \theta \cdot W + \psi * \tilde{\mu},$$

for some $\theta \in L^2_{\mathrm{loc}}(W)$ and $\psi \in \mathcal{G}_{\mathrm{loc}}(\mu)$, see Jacod and Shiryaev (2003).

For simplicity, we assume that the **numéraire** is a savings account:

$$X^0 = \exp\left(\int_0^\cdot r_s \,\mathrm{d}s\right),$$

with $r = (r_t)_{t\geq 0}$ representing the risk-free short rate (typically, OIS rate).

Reference: Fontana et al. (2020).

An alternative representation of FRA prices

Let us recall the model-free representation of FRA prices:

$$\Pi_t^{\mathrm{FRA}}(T,\delta,K) = \delta P_t(T+\delta)\big(L_t(T,T+\delta)-K\big),$$

which we rewrite as follows, using the notation $K(\delta) := 1+\delta K$:

$$\begin{aligned}\Pi_t^{\mathrm{FRA}}(T,\delta,K) &= P_t(T+\delta)\big(1+\delta L_t(T,T+\delta)\big) - K(\delta)P_t(T+\delta)\\ &= S_t^{\delta} P_t(T,\delta) - K(\delta)P_t(T+\delta),\end{aligned}$$

with

$$\boldsymbol{P_t(T,\delta)} := \frac{P_t(T+\delta)}{P_t(t+\delta)}\frac{1+\delta L_t(T,T+\delta)}{1+\delta L_t(t,t+\delta)}$$

and

$$\boldsymbol{S_t^{\delta}} := P_t(t+\delta)\big(1+\delta L_t(t,t+\delta)\big) =: \frac{1+\delta L_t(t,t+\delta)}{1+\delta L^{\mathrm{zcb}}(t,t+\delta)},$$

where L^{zcb} denotes the simple forward rate associated to risk-free ZCB bonds.

Terminology and interpretation:

1. S_t^{δ}: **spot multiplicative spread**, measures the relative riskiness of interbank rates with tenor δ at time t;
2. $P_t(T,\delta)$: δ-**tenor bond**, time-to-maturity behavior for tenor δ.

An alternative representation of FRA prices

These quantities admit a **foreign exchange analogy**:
let us imagine that a **foreign economy is associated to each tenor $\delta \in \mathcal{D}$**:

1. $P_t(T, \delta)$ represents the price of a ZCB of the foreign economy δ measured in units of the corresponding foreign currency;
2. S_t^δ represents the spot exchange rate between the foreign currency of economy δ and the domestic currency.

Then, the price of a foreign ZCB in units of the domestic currency is given by $S_t^\delta P_t(T, \delta)$ and the FRA becomes analogous to a FX forward contract.

Remark: this analogy suggests that this general HJM framework can be applied to other markets having multiple term structures, such as

- foreign exchange markets;
- energy markets;
- credit rating markets.

Remark: the classical single-curve setting corresponds to

$$S_t^\delta \equiv 1 \qquad \text{and} \qquad P_t(T, \delta) = P_t(T).$$

A general multi-curve HJM framework

We adopt the parametrization in terms of S_t^δ and $P_t(T,\delta)$ and suppose that

$$S_t^\delta = S_0^\delta \, \mathcal{E}\left(\int_0^\cdot \alpha_s^\delta \, \mathrm{d}s + \int_0^\cdot H_s^\delta \mathrm{d}W_s + \int_0^\cdot \int_E L^\delta(s,x)\tilde{\mu}(\mathrm{d}s,\mathrm{d}x)\right)$$

and, for all $\delta \in \mathcal{D}_0$ and $0 \leq t \leq T < +\infty$,

$$P_t(T,\delta) = \exp\left(-\int_t^T f_t(u,\delta)\,\mathrm{d}u\right),$$

where

$$\begin{aligned} f_t(T,\delta) = f_0(T,\delta) + \int_0^t a(s,T,\delta)\mathrm{d}s + \int_0^t b(s,T,\delta)\mathrm{d}W_s \\ + \int_0^t \int_E g(s,x,T,\delta)\tilde{\mu}(\mathrm{d}s,\mathrm{d}x). \end{aligned}$$

Technical assumptions: suitable integrability assumptions that ensure the applicability of ordinary and stochastic Fubini theorems to develop $\int_t^T f_t(u,\delta)\,\mathrm{d}u$.
(see Assumption 3.3 in Fontana et al. (2020) for details)

A general multi-curve HJM framework

Let us introduce the following notation, for all $0 \leq t \leq T$, $\delta \in \mathcal{D}_0$ and $x \in E$:

$$\bar{a}(t,T,\delta) := \int_t^T a(t,u,\delta)\mathrm{d}u, \;\; \bar{b}(t,T,\delta) := \int_t^T b(t,u,\delta)\mathrm{d}u, \;\; \bar{g}(t,x,T,\delta) := \int_t^T g(t,x,u,\delta)\mathrm{d}u.$$

Lemma

For every $T \in \mathbb{R}_+$ and $\delta \in \mathcal{D}_0$, it holds that

$$\begin{aligned} P(T,\delta) = P_0(T,\delta)\,\mathcal{E}\Bigg(\int_0^{\cdot} f_s(s,\delta)\mathrm{d}s - \int_0^{\cdot} \bar{a}(s,T,\delta)\mathrm{d}s + \frac{1}{2}\int_0^{\cdot} |\bar{b}(s,T,\delta)|^2\mathrm{d}s \\ - \int_0^{\cdot} \bar{b}(s,T,\delta)\mathrm{d}W_s - \int_0^{\cdot}\int_E \bar{g}(s,x,T,\delta)\tilde{\mu}(\mathrm{d}s,\mathrm{d}x) \\ + \int_0^{\cdot}\int_E \big(e^{-\bar{g}(s,x,T,\delta)} - 1 + \bar{g}(s,x,T,\delta)\big)\mu(\mathrm{d}s,\mathrm{d}x)\Bigg). \end{aligned}$$

By martingale representation, every density process $Z = (Z_t)_{t\geq 0}$ can be written as

$$Z = \mathcal{E}\big(-\theta\cdot W - \psi * \tilde{\mu}\big),$$

for some $\theta \in L^2_{\mathrm{loc}}(W)$ and $\psi : \Omega \times \mathbb{R}_+ \times E \to (-\infty, 1)$ belonging to $\mathcal{G}_{\mathrm{loc}}(\mu)$.

objective: characterize when Z is the density process of an ELMM Q.

A general multi-curve HJM framework

Let us define

$$\Lambda^*(t,x,T,\delta) := \big(1-\psi(t,x)\big)\big((1+L^\delta(t,x))e^{-\bar{g}(t,x,T,\delta)}-1\big) - L^\delta(t,x) + \bar{g}(t,x,T,\delta).$$

Proposition

Let $Q \sim P$ be a probability measure with density process Z represented as above. Then, Q is an ELMM if and only if, for all $T>0$,

$$\int_0^T \int_E |\Lambda^*(s,x,T,\delta)|\lambda_s(\mathrm{d}x)\mathrm{d}s < +\infty \text{ a.s.},$$

and the following two conditions hold a.s.

1. for a.e. $t \in \mathbb{R}_+$, it holds that

$$r_t = f_t(t,0),$$
$$\alpha_t^\delta = f_t(t,0) - f_t(t,\delta) + \theta_t^\top H_t^\delta + \int_E \psi(t,x) L^\delta(t,x)\lambda_t(\mathrm{d}x);$$

A general multi-curve HJM framework

Proposition (cont.)

2. for every $T > 0$ and for a.e. $t \in [0, T]$, it holds that

$$\bar{a}(t, T, \delta) = \frac{1}{2}|\bar{b}(t, T, \delta)|^2 + \bar{b}(t, T, \delta)^\top(\theta_t - H_t^\delta) + \int_E \Big((1 - \psi(t, x))(1 + L^\delta(t, x))(e^{-\bar{g}(t,x,T,\delta)} - 1) + \bar{g}(t, x, T, \delta)\Big)\lambda_t(\mathrm{d}x).$$

Proof (sketch):

- using the preceding Lemma and Yor's formula, write $Z(X^0)^{-1}S^\delta P(T, \delta)$ as a stochastic exponential $\mathcal{E}(Y)$, where the process Y can be explicitly computed;
- $\mathcal{E}(Y) \in \mathcal{M}_{\mathrm{loc}} \Longleftrightarrow Y \in \mathcal{M}_{\mathrm{loc}}$;
- $Y \in \mathcal{M}_{\mathrm{loc}}$ is equivalent to
  - Y has finite variation terms of locally integrable variation,
  - the predictable compensator Y^p of Y must be null;
- deduce that $Y^p \equiv 0 \Longleftrightarrow$ HJM conditions (1)-(2).

Reference: follows from a more general result in Fontana et al. (2020).

A general multi-curve HJM framework

Interpretation:

1. condition (1) means the following:
   - the instantaneous yield on a ZCB must equal the risk-free short rate r_t;
   - the instantaneous yield on the floating leg of a FRA must equal the instantaneous risk-free return r_t plus a risk premium term.
2. condition (2) is a generalization of the HJM drift restriction.

Remark: conditions (1)-(2) actually characterize ELMDs, i.e., all strictly positive $Z \in \mathcal{M}_{\mathrm{loc}}$ such that

$$Z\,(X^0)^{-1} S^{\delta} P(T,\delta)$$

is a local martingale, for all $(T,\delta) \in \mathbb{R}_+ \times \mathcal{D}_0$, with $S^0 \equiv 1$ and $P(T,0) := P(T)$. Therefore, the Proposition can be used to deduce explicit conditions guaranteeing NUPBR for the multi-curve market.

A hybrid LMM-HJM framework

In the spirit of **Libor market models (LMM)**, let us denote for each $\delta \in \mathcal{D}$:

- $\mathcal{T}^\delta = \{T_0^\delta, \ldots, T_{N^\delta}^\delta\}$ the set of settlement dates of traded FRAs with tenor δ;
- we assume that $T_0^\delta = T_0$ and $T_{N^\delta}^\delta = T^*$, for all $\delta \in \mathcal{D}$, for $T^* \in (0, +\infty)$;
- equidistant tenor structures: $T_i^\delta - T_{i-1}^\delta = \delta$, for all $i = 1, \ldots, N^\delta$;
- $\mathcal{T} := \bigcup_{\delta \in \mathcal{D}} \mathcal{T}^\delta$, corresponding to the set of traded FRAs;
- ZCBs are traded for all maturities in the set $\mathcal{T}^0 := \mathcal{T} \cup \{T^* + \delta; \delta \in \mathcal{D}\}$.

Under the above structure, we are considering finitely many traded assets.

In the spirit of LMM, we postulate **dynamics directly for the forward Libor rates**, for every $\delta \in \mathcal{D}$ and $T \in \mathcal{T}^\delta$:

$$L_t(T, T+\delta) = L_0(T, T+\delta) + \int_0^t a^L(s, T, \delta)\mathrm{d}s + \int_0^t b^L(s, T, \delta)\mathrm{d}W_s + \int_0^t \int_E g^L(s, x, T, \delta)\tilde{\mu}(\mathrm{d}s, \mathrm{d}x),$$

for $b^L(\cdot, T, \delta) \in L^2_{\mathrm{loc}}(W)$ and $g^L(\cdot, \cdot, T, \delta) \in \mathcal{G}_{\mathrm{loc}}(\mu)$.

A hybrid LMM-HJM framework

Proposition

Suppose that the conditions of the previous Proposition are satisfied for $\delta = 0$ and for all $T \in \mathcal{T}^0$. Let Q be a probability measure with density process Z as represented above. Then, Q is an ELMM for all traded FRAs if and only if

$$\int_0^T \int_E \left| g^L(s,x,T,\delta)\left((1-\psi(s,x))e^{-\bar{g}(s,x,T+\delta,0)} - 1\right)\right| \lambda_s(\mathrm{d}x)\mathrm{d}s < +\infty \text{ a.s.,}$$

and the following condition holds a.s., for all $\delta \in \mathcal{D}$, $T \in \mathcal{T}^\delta$ and a.e. $t \in [0,T]$:

$$\begin{aligned} a^L(t,T,\delta) &= b^L(t,T,\delta)^\top\left(\theta_t + \bar{b}(t,T+\delta,0)\right) \\ &\quad - \int_E g^L(t,x,T,\delta)\left((1-\psi(t,x))e^{-\bar{g}(t,x,T+\delta,0)} - 1\right)\lambda_t(\mathrm{d}x). \end{aligned}$$

Proof (sketch):

- the assumptions imply that $Z\,(X^0)^{-1}P(T+\delta)$ is a local martingale;
- apply the product rule to $L(T,T+\delta)Z\,(X^0)^{-1}P(T+\delta)$;
- apply similar reasoning as in the previous Proposition to characterize the local martingale property by analysing the finite variation terms.

Björk, T. (2004), On the geometry of interest rate models, in Carmona, R. et al. (eds.), *Paris-Princeton Lectures on Mathematical Finance 2003*, pp. 133-215, Springer.

Björk, T. (2020), *Arbitrage Theory in Continuous Time*, 4th ed., Oxford University Press.

Bianchetti, M. and Morini, M. (eds.) (2013), *Interest Rate Modeling after the Financial Crisis*, Risk Books.

Brigo, D. and Mercurio, F. (2006), *Interest Rate Models: Theory and Practice*, 2nd ed., Springer.

Cherny, A. and Shiryaev, A. (2005), On stochastic integrals up to infinity and predictable criteria for integrability, in Émery, M. et al. (eds.), *Séminaire de Probabilités XXXVIII*, pp. 165–185, Springer.

Cuchiero, C., Fontana, C. and Gnoatto, A. (2016), A general HJM framework for multiple yield curve modeling, *Finance and Stochastics*, 20(2): 267–320.

Cuchiero, C., Klein, I. and Teichmann, J. (2016), A new perspective on the fundamental theorem of asset pricing for large financial markets, *Theory of Probability and its Applications*, 60: 561–579.

Eberlein, E., Gerhart, C. and Grbac, Z. (2020), Multiple curve Lévy forward price model allowing for negative interest rates, *Mathematical Finance*, 30(1): 167–195.

Eberlein, E. and Raible, S. (1999), Term structure models driven by general Lévy processes, *Mathematical Finance*, 9(1): 31–53.

Flesaker, B. and Hughston, L. (1996), Positive interest, *Risk Magazine*.

Fontana, C., Grbac, Z., Gümbel, S. and Schmidt, T. (2020), Term structure modeling for multiple curves with stochastic discontinuities, *Finance and Stochastics*, 24: 465–511.

Grbac, Z. and Runggaldier, W. (2015), *Interest Rate Modeling: Post-Crisis Challenges and Approaches*, Springer.

Henrard, M. (2014), *Interest Rate Modelling in the Multi-curve Framework*, Palgrave Macmillan.

Jacod, J. and Shiryaev, A. (2003), *Limit Theorems for Stochastic Processes*, 2nd ed., Springer.

Jiao, Y., Ma, C. and Scotti, S. (2017(, Alpha-CIR model with branching processes in sovereign interest rate modeling, *Finance and Stochastics*, 21: 789–813.

Kabanov, Y., Kardaras, C. and Song, S. (2016), No arbitrage of the first kind and local martingale numéraires, *Finance and Stochastics*, 20: 1097–1108.

Karatzas, I. and Kardaras, C. (2007), The numéraire portfolio in semimartingale financial models, *Finance and Stochastics*, 11: 447–493.

Kardaras, C. (2013), On the closure in the Émery topology of semimartingale wealth-process sets, *Annals of Applied Probability*, 23(4): 1355–1376.

Musiela, M. and Rutkowski, M. (2005), *Martingale Methods in Financial Modelling*, 2nd ed., Springer.

Rogers, L.C.G. (1997), The potential approach to the term structure of interest rates and foreign exchanges, *Mathematical Finance*, 7(2): 157–176.