From Diamonds to Signatures

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Part I

Based on F-Gatheral-Radoičić, “Forests, cumulants, martingales.” Ann. Probab. 2022

Diamonds. Filtered $\mathbf{P}$-space, all martingales continuous, A_T a $\mathcal{F}_T$-measurable r.v.

$$X_t := \log \mathbf{E}_t e^{A_T} \, (\text{note: } X_T = A_T)$$

Gatheral and coworkers, 2017/2020: (formal) diamond expansion

$$\mathbf{E}_t e^{zX_T} = e^{zX_t + \frac{1}{2} z(z-1)(X \diamond X)_t(T) + \sum_{n \geqslant 2} \mathbb{F}_t^n(z;T)}$$

Def: For semimartingales X, X' on $[0,T]$, with $\langle X, X' \rangle_T \in L^1$, diamond product given by

$$(X \diamond X')_t(T) := \mathbf{E}_t \langle X, X' \rangle_{t,T} = \mathbf{E}_t \langle X, X' \rangle_T - \langle X, X' \rangle_t$$

Note: $\log \mathbf{E}_t e^{zX_T}$=: (conditional) cumulant generating function where terms $\mathbb{F}_t^n(z;T)$ satisfies a recursion.

Define $Y_t := \mathbf{E}_t A_T \quad (\text{note: } Y_T = A_T)$.

Thm: [FGR'22] Under natural integrability assumptions, for a, b small enough

$$\mathbf{E}_t e^{aY_T + b\langle Y\rangle_T} = e^{aY_t + b\langle Y\rangle_t + \sum_{n\geqslant 2}\mathbb{G}_t^n(a,b;T)}$$

with $\mathbb{G}^2 = \left(\frac{1}{2}a^2 + b\right)(Y \diamond Y)$ and recursion $\mathbb{G}^n = \frac{1}{2}\sum_{i=2}^{n-2}\mathbb{G}^{n-i} \diamond \mathbb{G}^i + aY \diamond \mathbb{G}^{n-1}$

Special cases: (i) $\frac{1}{2}a^2 + b = 0$ (exponential martingale case) $\Rightarrow$ corrector $\mathbb{G}$ vanishes

(ii) $b + a/2 = 0$, (rigorous) form of Alos et al. [AFR'20], since

$$\mathbf{E}_t e^{A_T} = e^{X_t} \equiv e^{Y_t - \langle Y\rangle_t/2} = \mathcal{E}(Y)_t, \quad Y = \mathcal{L}(e^X)$$

(iii) $b = 0$, Lacoin-Rohdes-Vargas '23

Many applications! (Bessel identities, Levy's area formula, rough forward variance models...)

Proof: (Sketch) For generic (continuous) semimartingale Z, sufficiently integrable, set

$$\Lambda_t^T := \log \mathbf{E}_t e^{Z_{t,T}} \quad \Leftrightarrow \quad \mathbf{E}_t e^{Z_T} =: e^{Z_t + \Lambda_t^T}$$

Trivially, the r.h.s is a martingale, considering its stochastic logarithm gives

$$\rightsquigarrow \quad \Lambda_t^T = \mathbf{E}_t \left(Z_{t,T} + \frac{1}{2} \langle Z + \Lambda^T \rangle_{t,T} \right) = \mathbf{E}_t Z_{t,T} + \frac{1}{2} (Z + \Lambda^T)_t^{\diamond 2}(T)$$

Fix a, b. Apply to $Z(\lambda) = \lambda a Y_T + \lambda^2 b \langle Y \rangle_T$. Note analyticity of $\lambda \mapsto \Lambda_t^T(\lambda)$ near 0, matching powers of λ leads to stated recursion.

Markovian perspective on diamond expansion

$X\ldots$ Markov diffusion with generator L. Recall (Feynman-Kac)

$h(t,x) := \mathbf{E}^{t,x} e^{\lambda(\varphi(X_T)+\int_t^T \xi(s,X_s)ds)}$, satisfies $(-\partial_t - L)h = \lambda h \xi, \quad h(T,\cdot) = e^{\lambda\varphi}$.

Cole-Hopf $h \equiv e^{\lambda v}$: With *carre du champ* operator, $2\Gamma(f) := L(f^2) - 2fLf$

$$L(\psi(f)) = \psi'(f)Lf + \psi''(f)\Gamma(f), \qquad L(e^{\lambda v}) = e^{\lambda v}(\lambda L v + \lambda^2 \Gamma(v)).$$

Obtain a HJB equation with small ($\rightsquigarrow$perturbative expansion) quadratic non-linearity

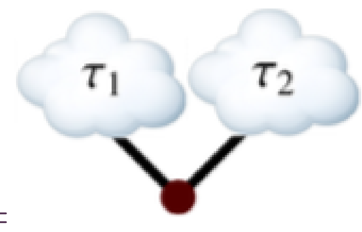
$$(-\partial_t - L)v = \lambda\Gamma(v) + \xi \quad v(T,\cdot) = \varphi.$$

Example ("KPZ with smooth noise") $L = \partial_x^2$. Then $\Gamma(f) := |\partial_x f|^2$.

Perturbative expansion of $\lambda v = \log h$ leads to (“Wild expansion”, as in Hairer’s KPZ paper)

$\lambda v = \lambda v(t,x) = \lambda u^{\bullet} + \lambda^2 u$  $+ \lambda^3 2u$  $+ \lambda^4 \Big(u$  $+ 4u$  $\Big) + \cdots$

$$= \sum_{|\tau| \geqslant 1} \lambda^{|\tau|} u^{\tau} = \sum_{n \geqslant 1} \lambda^n \sum_{\tau : |\tau| = n} u^{\tau} =: \sum_{n \geqslant 1} \lambda^n \mathbb{K}^n$$

with $u^{\tau} = K \star ((\partial_x u^{\tau_1})(\partial_x u^{\tau_2}))$, binary trees $\tau = [\tau_1, \tau_2] =$  and $|\tau| = \#\{\text{leaves}\}$.

Since every (binary) tree τ with $|\tau| = n+1$ leaves is of form $\tau = [\tau_1, \tau_2]$, we deduce with middle summation below over all trees τ_1, τ_2 with $|\tau_1| + |\tau_2| = n+1$,

$$\mathbb{K}^{n+1} = \sum_{\tau : |\tau| = n+1} u^{\tau} = \sum_{\cdots} u^{[\tau_1, \tau_2]} = \dots . = \frac{1}{2} \sum_{i=1}^{n} \mathbb{K}^{i} \diamond \mathbb{K}^{n+1-i}$$

which is the special case $b = 0$ of the diamond expansion.

Message: Cumulants in Markovian setting described by HJB / KPZ type PDEs.

PS: Gaussian perspective on diamond expansion: consistent with Nourdin–Peccati (JFA '10)

Part II

Based on F-Hager-Tapia. "Unified signature cumulants and generalized Magnus expansions." Forum Math. Sigma 2022

Let $X: [0,T] \to \mathbf{R}^d$, smooth, indefinite signature of X given by

$$S_t = \mathrm{Sig}(X)_t = \left(1, \int_0^t dX, \int_0^t \int_0^s dX \star dX, \ldots\right) \in \mathbf{R} \oplus \mathbf{R}^d \oplus (\mathbf{R}^d)^{\otimes 2} \oplus \ldots =: T((\mathbf{R}^d))$$

satifies linear differential equations in $\mathcal{T} = (T((\mathbf{R}^d)), +, \star)$, with tensor (concatenation) product,

$$dS_t = S_t \star dX_t, \quad S_0 = \mathbf{1} = (1, 0, 0, \ldots) \in \mathcal{T}_1$$

Power series calculus! In particular: $\exp_\star : \mathcal{T}_0 \to \mathcal{T}_1$ with inverse $\log_\star : \mathcal{T}_1 \to \mathcal{T}_0$

Drop all $\star$'s in what follows / introduce commutator bracket $[a,b] = ab - ba$

Log-signature of X given by

$$L_t := \log S_t = \ldots = \left(0, \int_0^t dX, \frac{1}{2}\int_0^t \int_0^s [dX, dX], \ldots \right) \in \mathcal{T}_0 \cong \mathbf{R}^d \oplus (\mathbf{R}^d)^{\otimes 2} \oplus \ldots$$

Differential evolution: with $S_t = \exp(L_t)$

$$dS_t = S_t \, dX_t \quad \Leftrightarrow \quad e^{-L_t} \partial_t e^{L_t} = \dot{X}_t$$

Rmk: In commutative setting have $\dot{L} = e^{-L_t} \partial_t e^{L_t}$, conclude $L_t = \int_0^t dX \quad \Leftrightarrow \quad S_t = e^{\int_0^t dX}$.

Theorem (Hausdorff 1906) For explicitly computable $G(z) = 1 + g_1 z + g_2 z^2 + \ldots$ have

$$\dot{X}_t = e^{-L_t}\partial_t e^{L_t} = G(\mathrm{ad}\, L_t)\dot{L_t} := \dot{L} + g_1[L, \dot{L}] + g_2[L, [L, \dot{L}]] + \cdots$$

Proof: Classical, in this setting see e.g. F-Victoir CUP '10

With $H(z) := 1/G(z) = 1 + h_1 z + h_2 z^2 + \cdots$ [with $h_j = B_j/j!$ Bernoulli numbers, $B_1 = -1/2$]

$$\dot{L} = H(\mathrm{ad}\, L)\dot{X} = \dot{X} + h_1[L, \dot{X}] + h_2[L, [L, \dot{X}]] + \cdots$$

Computing L **by recursion** (a.k.a. Magnus expansion). Follows from

$$L = (0, L^1, L^2, \dots) \in \mathcal{T}^{\geqslant 1}, \quad X \equiv (0, X, 0, 0, \dots) \in \mathcal{T}^{=1}$$

and

$$\dot{L} = H(\mathrm{ad}\, L)\dot{X} = \dot{X} + h_1[L, \dot{X}] + h_2[L, [L, \dot{X}]] + \cdots$$

Explicit: $L_t^1 = \int_0^t dX$, $L_t^2 = -\frac{1}{2}\int_0^t \int_0^s [dX_r, dX_s]$ ... and with general term (e.g. Wiki)

$$L_t^n = \sum_{k=1}^{n-1} \frac{B_k}{k!} \sum_{|\ell|=k, \|\ell\|=n-1,} \int_0^t \mathrm{ad}_{L_s^{l_1}} \circ \dots \circ \mathrm{ad}_{L_s^{l_k}} d X_s,$$

$$\ell = (l_1, \dots, l_k),\ l_i \geq 1,\ |\ell| = k,\ ||\ell|| = l_1 + \dots + l_k$$

Why good idea?

respect geometry: $e^L = e^{\sum L^i} \approx e^{L^1 + \cdots + L^N}$ still grouplike

sparsity: e.g. $v = (0, v, 0, 0, \ldots) \in \mathcal{T}^{=1}$ vs. $e^v = (1, v, v^2/2, v^3/3!, \ldots) \in \mathcal{T}^{''\text{full}''}$

"*ultimate simplification, new insight, and superior computational algorithms*" [A. Iserles]

www.ams.org › notices › fea-iserles - Diese Seite übersetzen

Expansions That Grow on Trees - American Mathematical ...

Expansions That **Grow** on **Trees**. Arieh **Iserles**. 430. NOTICES OF THE AMS. VOLUME 49, NUMBER 4. Linear Ordinary Differential Equations. How to solve ...

von A Iserles - 2002 - Zitiert von: 53 - Ähnliche Artikel

arxiv.org › math-ph ▾ Diese Seite übersetzen

The Magnus expansion and some of its applications

30.10.2008 - When formulated in operator or matrix form, the **Magnus expansion** furnishes an elegant setting to built up approximate exponential ...

von S Blanes - 2008 - Zitiert von: 876 - Ähnliche Artikel

Expected signatures (T. Lyons and many)

$X\colon [0,T]\times\Omega\to\mathbf{R}^d$ ... (sufficiently integrable) continuous semimartingale

Stratonovich indefinite signature of X given by

$$dS = S\circ dX\,,\quad S_0 = \mathbf{1} = (1,0,0,...)\in\mathcal{T}$$

Expected signature given by $\boldsymbol{\mu}_T := \mathbf{E}S_T\in\mathcal{T}$

Bonnier-Oberhauser '19 study signature cumulants

$$\boldsymbol{\kappa}_T := \log_\star\boldsymbol{\mu}_T \quad\Leftrightarrow\quad \boldsymbol{\mu}_T = \exp_\star\boldsymbol{\kappa}_T$$

NB: we are back in $(T((\mathbf{R}^d)),+,\star)=\mathcal{T}$ [and drop again $\star$'s in what follows]

Example (Time-inhomogenous Brownian motion). Let $dX_t = \sigma(t) dB_t$. Then

$$dS_t = S_t \circ dX_t = (....)dX + \tfrac{1}{2} S_t d\langle X, X\rangle_t = (...)dB + S_t a(t) dt$$

with covariance matrix of X_t given by $a(t) = \sigma\sigma^T(t)/2$. With $\boldsymbol{\mu}_t := \mathbf{E} S_t \in \mathcal{T}$ as before, get

$$d\boldsymbol{\mu}_t = \boldsymbol{\mu}_t a(t) dt$$

This is a linear ODE in $\mathcal{T}$, with $a(t) \equiv (0, 0, a(t), 0, 0, \ldots) \in \mathcal{T}^{=2}$.

We then get the following Magnus expansion for signature cumulants

$$\boldsymbol{\kappa}_T := \log_\star \boldsymbol{\mu}_T = \left(\int_0^T a(t) dt - \frac{1}{2} \int_0^T \left[\int_0^t a(s) ds, a(t)\, dt \right] + \cdots \right)$$

Example (cont'd). Assume $a(t) \equiv \frac{1}{2}\mathrm{Id}$ i.e. X is standard Brownian motion in $\mathbf{R}^d$

Then all commutators vanish and we recover Fawcett's formula

$$\boldsymbol{\kappa}_T(X) = \frac{T}{2} \times \mathrm{Id} \qquad \Leftrightarrow \qquad \mathbf{E}\mathrm{Sig}(B)_T = \exp_\star\left(\frac{T}{2} \times \mathrm{Id}\right)$$

The unified functional equation (FE)

$X\colon [0,T]\times\Omega\to\mathbf{R}^d$ ... (sufficiently integrable) continuous semimartingale.

Thm [F-Hager-Tapia '22] With $\boldsymbol{\kappa}_t := \boldsymbol{\kappa}_t^T := \log \mathbf{E}_t \mathrm{Sig}(Z|_{[t,T]})$ we have

$$\boldsymbol{\kappa}_t^T = \mathbf{E}_t\{(1)+(2)+(3)+(4)\},$$

$$
\begin{aligned}
(1) &= \int_t^T H(\mathrm{ad}\,\boldsymbol{\kappa}_u)\,dX_u\\
(2) &= \frac{1}{2}\int_t^T H(\mathrm{ad}\,\boldsymbol{\kappa}_u)\,d\langle X\rangle_u\\
(3) &= \int_t^T H(\mathrm{ad}\,\boldsymbol{\kappa}_u)\circ(\mathrm{Id}\odot Q)(\mathrm{ad}\,\boldsymbol{\kappa}_u)\,d[\![X,\boldsymbol{\kappa}]\!]_u\\
(4) &= \frac{1}{2}\int_t^T H(\mathrm{ad}\,\boldsymbol{\kappa}_u)\circ Q(\mathrm{ad}\,\boldsymbol{\kappa}_u)\,d[\![\boldsymbol{\kappa},\boldsymbol{\kappa}]\!]_u
\end{aligned}
$$

Compare with FE underlying diamonds expansions: $X \rightsquigarrow Z\colon [0,T]\times\Omega\to\mathbf{R}$

$$\Lambda_t^T = \log \mathbf{E}_t e^{Z_{t,T}} = \mathbf{E}_t\left(Z_{t,T} + \frac{1}{2}\langle Z+\Lambda^T\rangle_{t,T}\right) = \mathbf{E}_t\left(Z_{t,T} + \frac{1}{2}\langle Z\rangle_{t,T} + \langle Z,\Lambda^T\rangle_{t,T} + \frac{1}{2}\langle \Lambda^T\rangle_{t,T}\right)$$

with e.g. $\langle Z^i, \boldsymbol{\kappa}^{jk}\rangle \mathfrak{e}_{ijk} \in \mathcal{T}^{=3}$, but $[\![\boldsymbol{\kappa}^{ij}, \boldsymbol{\kappa}^{klm}]\!] \mathfrak{e}_{ij} \tilde{\otimes} \mathfrak{e}_{klm} \in \mathcal{T}^{=2} \tilde{\otimes} \mathcal{T}^{=3}$, and

$$G(\mathrm{ad}_x) = \sum_{k=0}^{\infty} \frac{(\mathrm{ad}_x)^k}{(k+1)!} \quad \textit{and} \quad Q(\mathrm{ad}_x) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} 2 \frac{(\mathrm{ad}_x)^n \odot (\mathrm{ad}_x)^{m}}{(n+1)!(m)!(n+m}$$

Note: $G(0) = \mathrm{Id}$, $\quad Q(0) = \mathrm{Id} \odot \mathrm{Id}$, with $(f \odot g)(a, b) = f(a) \star f(b)$, for $f, g \colon \mathcal{T} \to \mathcal{T}$

Corollary: For $\hat{\boldsymbol{\kappa}}_t^T = \mathrm{Sym}(\boldsymbol{\kappa}_t^T)$, the ($t$-conditional) multivariate cumulants of $Z_{t,T}$, we find the "diamond" functional equation, but now in $\mathrm{Sym}((\mathbf{R}^d)) = T((\mathbf{R}^d)) / \sim$.

$$\hat{\boldsymbol{\kappa}}_t^T = \mathbf{E}_t\left(Z_{t,T} + \frac{1}{2}\langle Z\rangle_{t,T} + \langle Z, \hat{\boldsymbol{\kappa}}^T\rangle_{t,T} + \frac{1}{2}\langle \hat{\boldsymbol{\kappa}}^T\rangle_{t,T} \right) = \mathbf{E}_t\left(Z_{t,T} + \frac{1}{2}(Z + \hat{\boldsymbol{\kappa}}^T)^{\diamond 2}_{t,T} \right)$$

with diamond product extended to $\mathrm{Sym}((\mathbf{R}^d))$-valued semimartingales.

Corollary: Apply to $Z(t,w) = X(t)$ for a smooth path $X\colon [0,T] \to \mathbf{R}^d$.

Can drop all $\mathbf{E}_t$ and all brackets, and recover (backward) Magnus, with

$$\boldsymbol{\kappa}_t = Z_{t,T} + \int_t^T (G - \mathrm{Id})(\mathrm{ad}_{\boldsymbol{\kappa}}) d\boldsymbol{\kappa} \Rightarrow -\dot{Z} = G(\mathrm{ad}_{\boldsymbol{\kappa}})\dot{\boldsymbol{\kappa}} \Leftrightarrow -\dot{\boldsymbol{\kappa}} = H(\mathrm{ad}_{\boldsymbol{\kappa}})\dot{Z}$$

Important remark: Our unified functional equation comes with a natural recursions / expansion, which provides a common generalization of Magnus - and diamond expansions.

Rmk (Exercise): apply general theorem to recover $\boldsymbol{\kappa}_t := \log \mathbf{E}\mathrm{Sig}(\int_0^\cdot \sigma(t) dB_t)$.

Concluding remarks

- In Markovian situation: computing $\mathbf{E}_t(\dots)$ is solving a (backward) PDE.

  Ni,Lyons ('15) Expected signature $\boldsymbol{\mu}$ of Markov diffusion (at time T), Brownian motion stopped at some ∂ ...

  In essence: $\boldsymbol{\mu} = (1, \boldsymbol{\mu}^1, \boldsymbol{\mu}^2, \dots)$ satisfies triangular system of linear PDEs (parabolic resp. elliptic, backward). Solved recursively,

$$\boldsymbol{\mu}_t^n = \Phi(\boldsymbol{\mu}_\tau^1, \dots \boldsymbol{\mu}_\tau^{n-1} : t \leqslant \tau \leqslant T)$$

  Signature cumulants $\log \boldsymbol{\mu}_t = \boldsymbol{\kappa}_t$ satisfies system of non-linear PDEs of KPZ type in $\mathcal{T}$

- Extension to (cadlag) semimartingales possible (F-Hager-Tapia'22), cumulant case also considered by Fukasawa-Matsushita '21.

Thank you!