Quantum Stochastic Processes for Quantum Computing

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Abstract. These lectures introduce stochastic models for noisy quantum dynamics with applications to quantum computing. We review stochastic Schrödinger equations and Lindblad dynamics as effective descriptions of open quantum systems, and illustrate their use in analyzing single-qubit gate fidelity. To model genuinely quantum noise, we introduce bosonic Fock space and develop the Hudson–Parthasarathy theory of quantum stochastic differential equations, providing a unitary dilation of Markovian open-system evolution with general jump operators. The theory is illustrated through explicit qubit examples, including dephasing and amplitude damping, with an emphasis on conceptual clarity.

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1 Classical noncommutative processes

As we have seen, a quantum circuit is essentially a noisy, controlled Schrödinger evolution. From a physical perspective, this noise arises from the unavoidable interaction between the system of interest and its environment. As a result, realistic quantum devices cannot be modeled as closed systems evolving unitarily on a Hilbert space, but rather as *open quantum systems*, whose effective dynamics are irreversible.

Throughout these notes, we denote by $\mathcal{H}$ a complex Hilbert space and $\mathscr{B}(\mathcal{H})$ the space of bounded operators on $\mathcal{H}$. We further consider the following subspaces of $\mathscr{B}(\mathcal{H})$ that play a distinguishing role:

$$\begin{aligned}
\mathscr{U}(\mathcal{H}) &:= \{\mathsf{a} \in \mathscr{B}(\mathcal{H}) : \mathsf{a}^\dagger \mathsf{a} = \mathsf{I}_{\mathcal{H}}\} && \textit{(unitary operators)}\\
\mathscr{O}(\mathcal{H}) &:= \{\mathsf{a} \in \mathscr{B}(\mathcal{H}) : \mathsf{a}^\dagger = \mathsf{a}\} && \textit{(observables)}\\
\mathscr{D}(\mathcal{H}) &:= \{\mathsf{a} \in \mathscr{O}(\mathcal{H}) : \mathsf{a} \succeq 0,\ \mathrm{tr}[\mathsf{a}] = 1\} && \textit{(density operators)}\\
\mathscr{P}(\mathcal{H}) &:= \{\mathsf{a} \in \mathscr{O}(\mathcal{H}) : \mathsf{a} = |\psi\rangle\langle\psi|,\ \|\psi\|_{\mathcal{H}} = 1\} && \textit{(pure states)}
\end{aligned}$$

where $\mathsf{a}^\dagger := \bar{\mathsf{a}}^\top$ is the hermitian conjugate of an operator $\mathsf{a} \in \mathscr{B}(\mathcal{H})$ and we will adopt the *bra-ket* notation to distinguish between primal vectors $|\psi\rangle$ (*ket*) and dual vectors $\langle\psi|$ (*bra*) for any vector $\psi \in \mathcal{H}$. Notice that $\mathscr{P}(\mathcal{H})$ is simply the subset of rank-1 projection operators.

1.1 Lindblad equation

A wide range of mathematical and physical models have been developed to describe open quantum systems, spanning microscopic Hamiltonian descriptions of system-environment interactions to phenomenological effective equations. Among these, one of the simplest and most widely used frameworks is provided by the *Lindblad (or Gorini-Kossakowski-Sudarshan-Lindblad, GKSL) equation.* Its importance stems from the fact that it provides the most general form of a Markovian, time-homogeneous quantum evolution compatible with the basic principles of quantum mechanics.

Rather than describing the system by a wave function, the Lindblad equation governs the evolution of the density operator $t \mapsto \rho_t \in \mathscr{D}(\mathcal{H})$, which encodes both classical and quantum uncertainty. The evolution with initial datum $\rho_0 \in \mathscr{D}(\mathcal{H})$ is given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho_t = -i[H, \rho_t] + \mathscr{L}^*(\rho_t), \qquad \mathscr{L}^*(\rho) := \tfrac{1}{2}\sum\nolimits_j \big(2L_j\rho L_j^\dagger - L_j^\dagger L_j \rho - \rho L_j^\dagger L_j\big), \tag{GKSL}$$

where $H \in \mathscr{O}(\mathcal{H})$ is a given system Hamiltonian, $\mathscr{L}^*$ is the *Lindblad (super)-operator*, and $L_j \in \mathscr{B}(\mathcal{H})$ are mutually orthogonal *jump operators*, describing the coupling to the environment. The first term represents coherent unitary evolution, while the second term captures irreversible processes such as decoherence, relaxation, and dissipation.

Remark 1.1 If $L_j \in \mathscr{O}(\mathcal{H})$ are observables, then the Lindblad operator takes the form

$$\mathscr{L}^*(\rho) = -\tfrac{1}{2}\sum\nolimits_j [L_j, [L_j, \rho]].$$

Remark 1.2 The use of * in $\mathscr{L}^*$ indicates that $\mathscr{L}^*$ is the adjoint of an operator $\mathscr{L}$ under an appropriate scalar product. In this case, the scalar product considered is the Hilbert-Schmidt scalar product $\langle \mathsf{a}, \mathsf{b}\rangle_{\mathscr{O}(\mathcal{H})} := \mathrm{tr}[\mathsf{a}^\dagger \mathsf{b}]$ since the objects in question are observables. Therefore,

$$\langle \mathscr{L}^*(\mathsf{a}), \mathsf{b}\rangle_{\mathscr{O}(\mathcal{H})} = \mathrm{tr}[(\mathscr{L}^*(\mathsf{a}))^\dagger \mathsf{b}] = \mathrm{tr}[\mathscr{L}^*(\mathsf{a})\mathsf{b}] = \mathrm{tr}[\mathsf{a}\mathscr{L}(\mathsf{b})] = \langle \mathsf{a}, \mathscr{L}(\mathsf{b})\rangle_{\mathscr{O}(\mathcal{H})},$$

with

$$\mathscr{L}(\mathsf{a}) = \tfrac{1}{2}\sum\nolimits_j \big(2L_j^\dagger \mathsf{a} L_j - L_j^\dagger L_j \mathsf{a} - \mathsf{a} L_j^\dagger L_j\big).$$

A thorough informal derivation of the Lindblad equation is lengthy and complex, so we will skip it here. However, from an operational point of view, the density operator ρ is an ensemble of pure states $\mathsf{a} = |\psi\rangle\langle\psi| \in \mathscr{P}(\mathcal{H})$. In other words, ρ can be interpreted as the expectation of the canonical random variable under a probability measure P on pure states $\mathscr{P}(\mathcal{H})$, i.e.,

$$\rho = \int_{\mathscr{P}(\mathcal{H})} \mathsf{a}\,\mathsf{P}(\mathsf{da}) \ \in \mathscr{D}(\mathcal{H}).$$

In the following, we will be interested in *stochastic dilations* or *stochastic unravellings* of the Lindblad equation, i.e., we will look at $\mathscr{P}(\mathcal{H})$-valued processes for which their expectation solves the Lindblad equation. The following sections provide examples of such processes, where information about the environment is embedded into the model.

1.2 Stochastic Schrödinger equation

In the context of Rydberg atoms, the optical control system is a primary source of noise. In the semiclassical limit of light-matter interactions, such noise sources can be considered classical. Without going into the details of the physics, we introduce the stochastic Schrödinger equation, which describes the evolution of a quantum system driven by classical noise sources.

Given an observable $L \in \mathscr{O}(\mathcal{H})$ and a real-valued smooth process β_t, the Schrödinger equation driven by the time-dependent observable $\dot{\beta}_t L \in \mathscr{O}(\mathcal{H})$ may be expressed as an evolution in the space of unitaries $\mathscr{U}(\mathcal{H})$:

$$\mathrm{d}U_t = iLU_t\,\mathrm{d}\beta_t, \qquad U_0 = \mathsf{I}_{\mathcal{H}}.$$

In this simple case, the solution may be explicitly expressed as

$$U_t = \exp\big(i(\beta_t - \beta_0)L\big)\mathsf{I}_{\mathcal{H}}.$$

Remark 1.3 When $\mathcal{H} = \mathbb{C}^n$, $\mathscr{U}(\mathcal{H})$ is a compact Lie group with Lie algebra

$$\mathscr{A}(\mathcal{H}) := \big\{\mathsf{a} \in \mathscr{B}(\mathcal{H}) : \mathsf{a}^\dagger = -\mathsf{a}\big\} = \big\{i\mathsf{a} \in \mathscr{B}(\mathcal{H}) : \mathsf{a} \in \mathscr{O}(\mathcal{H})\big\},$$

i.e., the space of skew-hermitian matrices. The Lie algebra $\mathscr{A}(\mathcal{H})$ can be equipped with a real inner product given by the Hilbert-Schmidt scalar product

$$\langle \mathsf{a}, \mathsf{b}\rangle_{\mathscr{A}(\mathcal{H})} := -\operatorname{tr}(\mathsf{a}^\dagger \mathsf{b}) = \Re\operatorname{tr}(\mathsf{a}\mathsf{b}^\dagger),$$

which is positive-definite on $\mathscr{A}(\mathcal{H})$. The associated norm is then

$$|\mathsf{a}|^2_{\mathscr{A}(n)} = \langle \mathsf{a}, \mathsf{a}\rangle_{\mathscr{A}(n)} = \operatorname{tr}(\mathsf{a}\mathsf{a}^\dagger).$$

Similarly, the space $\mathscr{O}(\mathcal{H})$ of observables may be equipped with the Hilbert-Schmidt scalar product $\langle \mathsf{a}, \mathsf{b}\rangle_{\mathscr{O}(\mathcal{H})} = \operatorname{tr}(\mathsf{a}^\dagger \mathsf{b}) \in \mathbb{R}$.

From a geometrical perspective, the unitary evolution is the exponential map applied to the time-dependent right-invariant vector field $\mathfrak{a}_t : \mathscr{U}(\mathcal{H}) \to \mathscr{A}(\mathcal{H})$, $\mathfrak{a}_t(U) = i\dot{\beta}_t HU = \mathfrak{a}_t(\mathsf{I}_{\mathcal{H}})U$. Throughout, we will consider right-multiplication, with U always acting on the right. $\diamondsuit$

In the presence of noise, however, β_t may no longer be smooth. Nevertheless, if $\beta_t = \omega_t$ is the Brownian motion, then the Itô formula applies and we get for $U_t := \exp(i\omega_t L)\mathsf{I}_{\mathcal{H}}$,

$$\mathrm{d}U_t = \big(iL\,\mathrm{d}\omega_t - \tfrac{1}{2}L^2\,\mathrm{d}t\big)U_t = iLU_t \circ \mathrm{d}\omega_t,$$

where $\circ\,\mathrm{d}\omega_t$ denotes the Stratonovich integral. This is precisely a stochastic Schrödinger equation with one noise operator L and one driving noise ω_t.

1.3 Unitary-valued processes

More generally, we consider a set $\mathscr{L} = \{H, L_1, \dots, L_d\} \subset \mathscr{O}(\mathcal{H})$ of orthonormal observables on the n-dimensional complex Hilbert space $\mathcal{H}$ under the Hilbert-Schmidt scalar product on $\mathscr{O}(\mathcal{H})$, where $1 \leq d \leq n^2$ is the number of noise channels. These observables will often be called noise operators. Further, let $\omega_t^1, \dots, \omega_t^d$ be independent standard real-valued Brownian motions and $\beta_t^1, \dots, \beta_t^d$ be Itô processes of the form

$$\beta_t^j = \beta_0^j + \mathsf{b}_t^j + \sqrt{\gamma_j}\,\omega_t^j, \qquad j = 1, \dots, d,$$

where b_t^j is an absolutely continuous process and $\gamma_j > 0$.

Define the $\mathscr{O}(\mathcal{H})$–valued (possibly degenerate) Brownian driver

$$\chi_t = -itH + \sum\nolimits_j iL_j\,\beta_t^j \quad \in \mathscr{A}(\mathcal{H}).$$

The intrinsic $\mathscr{U}(\mathcal{H})$-valued diffusion process solves the Stratonovich SDE

$$\mathrm{d}U_t = \circ\,\mathrm{d}\chi_t\,U_t, \qquad U_0 = \mathsf{I}_{\mathcal{H}}, \tag{SSE}$$

or in components

$$\mathrm{d}U_t = -iHU_t\,\mathrm{d}t + \sum\nolimits_j iL_jU_t\,\mathrm{d}\mathsf{b}_t^j + \sum\nolimits_j \sqrt{\gamma_j}\,iL_jU_t \circ \mathrm{d}\omega_t^j,$$

where H is a system Hamiltonian. The matrix-valued quadratic variation of χ is

$$\mathrm{d}\langle\chi\rangle_t = \sum\nolimits_j \gamma_j\,iL_j \otimes iL_j\,\mathrm{d}t.$$

If $d = n^2$, the driver is *elliptic* (non-degenerate). If $d < n^2$, the covariance has rank d and the process is *hypoelliptic* (degenerate), exploring only the connected subgroup

$$\exp(\mathscr{A}_{\mathscr{L}}) \subset \mathscr{U}(\mathcal{H}), \quad \mathscr{A}_{\mathscr{L}} = \mathrm{Lie}\{iH, iL_1, \dots, iL_r\} \subset \mathscr{A}(\mathcal{H}).$$

Clearly, there are situations where $\mathscr{A}_{\mathscr{L}} = \mathscr{A}(\mathcal{H})$ for $d < n^2$, in which case, the full group of unitaries is explored, i.e., $\exp(\mathscr{A}_{\mathscr{L}}) = \mathscr{U}(\mathcal{H})$.

In Itô form, the equivalent SDE reads

$$\mathrm{d}U_t = \big(\mathrm{d}\chi_t + \mathfrak{I}\,\mathrm{d}t\big)U_t, \qquad \mathfrak{I} := -\tfrac{1}{2}\sum\nolimits_j \gamma_j L_j^2,$$

where $\mathfrak{I}$ is the Laplace-Beltrami operator associated with the right-invariant connection.

Remark 1.4 When $H = 0$, $d = 1$, we find, as in the deterministic case, the explicit solution

$$U_t = \exp\big(i(\mathsf{b}_t + \sqrt{\gamma}\omega_t)L\big)\mathsf{I}_{\mathcal{H}}.$$

In particular, U_t commutes with L for all $t \geq 0$. $\diamondsuit$

Towards the Lindblad equation To obtain the Lindblad equation (GKSL) from the stochastic Schrödinger equation (SSE), we consider noise profiles of the form

$$\beta_t^j = \int_0^t u^j(r)\,\mathrm{d}r + \omega_t^j, \qquad j = 1, \dots, d,$$

Now let $\rho_0 \in \mathscr{D}(\mathcal{H})$ be an initial datum for the Lindblad equation and U_t be the solution to (SSE). Then, the $\mathscr{D}(\mathcal{H})$-valued process $\mathsf{a}_t \coloneqq U_t\,\rho_0\,U_t^\dagger$ satisfies

$$\begin{aligned}
\mathrm{d}\mathsf{a}_t &= \mathrm{d}U_t\,\rho_0\,U_t^\dagger + U_t\,\rho_0\,\mathrm{d}U_t^\dagger + \mathrm{d}U_t\,\rho_0\,\mathrm{d}U_t^\dagger \\
&= (\mathrm{d}\chi_t + (-iH + \mathfrak{I})\,\mathrm{d}t)\,\mathsf{a}_t + \mathsf{a}_t(\mathrm{d}\chi_t^\dagger + (-iH + \mathfrak{I})^\dagger \mathrm{d}t) + \mathrm{d}\chi_t\,\mathsf{a}_t\,\mathrm{d}\chi_t^\dagger \\
&= -i[H_t^u, \mathsf{a}_t]\,\mathrm{d}t + i\sum\nolimits_j [L_j, \mathsf{a}_t]\sqrt{\gamma_j}\,\mathrm{d}\omega_t^j + \tfrac{1}{2}\sum\nolimits_j \gamma_j\big(2L_j\,\mathsf{a}_t\,L_j - L_j^2\mathsf{a}_t - \mathsf{a}_t L_j^2\big)\,\mathrm{d}t \\
&= -i[H_t^u, \mathsf{a}_t]\,\mathrm{d}t + \mathscr{L}(\mathsf{a}_t)\,\mathrm{d}t + i\sum\nolimits_j \sqrt{\gamma_j}[L_j, \mathsf{a}_t]\,\mathrm{d}\omega_t^j,
\end{aligned}$$

where we set the Hamiltonian $H_t^u \coloneqq H - \sum_j u_j(t) L_j$ and

$$\mathscr{L}(\rho) = \tfrac{1}{2}\sum\nolimits_j \big(2\widehat{L}_j\,\rho\,\widehat{L}_j - \widehat{L}_j^2\rho - \rho\widehat{L}_j^2\big), \qquad \widehat{L}_j \coloneqq \sqrt{\gamma_j}L_j.$$

Hence, taking the expectation, we obtain for the density operator $\rho_t \coloneqq \mathbb{E}[\mathsf{a}_t] \in \mathscr{D}(\mathcal{H})$ the Lindblad equation

$$\mathrm{d}\rho_t = -i[H_t^u, \rho_t]\,\mathrm{d}t + \mathscr{L}^*(\rho_t)\,\mathrm{d}t,$$

where we used the fact that $\mathscr{L}^* = \mathscr{L}$ since $L_j \in \mathscr{O}(\mathcal{H})$.

Note, however, that we recover the Lindblad equation with *only* Hermitian jump operators in this way, i.e., $L_j \in \mathscr{O}(\mathcal{H})$. To consider general jump operators, we have to leave the realm of classical noise and talk about quantum noise, which will be the main topic of the remaining sections in this lecture series.

Remark 1.5 We note that if $\rho_0 \in \mathscr{P}(\mathcal{H})$ is a pure state, then a_t is a $\in \mathscr{P}(\mathcal{H})$-valued process, i.e., a_t is almost surely a pure state for all times. $\diamondsuit$

1.4 Example: 1-qubit fidelity of the Hadamard gate

In quantum computing, one is often interested in the *fidelity* of a quantum gate, i.e., a unitary operation, where the fidelity of two density operators $\rho, \sigma \in \mathscr{D}(\mathcal{H})$ is defined by

$$\mathsf{F}(\rho, \sigma) \coloneqq \mathrm{tr}\Big[(\sqrt{\rho}\sigma\sqrt{\rho})^{\frac{1}{2}}\Big]^2.$$

When, $\rho = |\psi\rangle\langle\psi| \in \mathscr{P}(\mathcal{H})$ is a pure state, then the fidelity reduces to

$$\mathsf{F}(\rho, \sigma) = \langle \psi, \sigma\psi\rangle = \mathrm{tr}[\rho\sigma].$$

If $\sigma = |\varphi\rangle\langle\varphi| \in \mathscr{P}(\mathcal{H})$ is also a pure state, then it simplifies to $\mathsf{F}(\rho, \sigma) = |\langle\psi, \varphi\rangle|^2$, which is much simpler than the case for general density operators. Let's see how the SSE can be used to compute the fidelity of a quantum gate.

Say, we would like to implement a Hadamard gate on a single qubit on the Hilbert space $\mathcal{H} = \mathbb{C}^2 = \mathrm{span}\{\mathbf{e}_0, \mathbf{e}_1\}$. The Hadamard gate is given by

$$U_{\mathsf{h}} = \frac{|\mathbf{e}_0\rangle + |\mathbf{e}_1\rangle}{\sqrt{2}}\langle\mathbf{e}_0| + \frac{|\mathbf{e}_0\rangle - |\mathbf{e}_1\rangle}{\sqrt{2}}\langle\mathbf{e}_1| = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix},$$

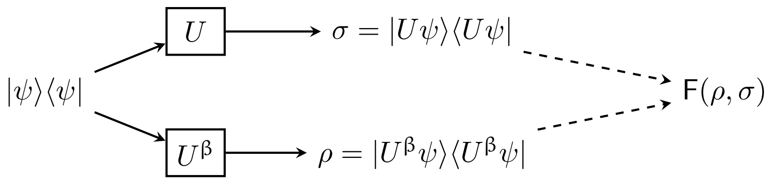

Figure 1: Computing the fidelity of a quantum gate

with the corresponding eigenpairs $(1, \mathbf{e}_+)$ and $(-1, \mathbf{e}_-)$. By spectral calculus, we obtain

$$\begin{aligned} U_{\mathsf{h}} &= (1)|\mathbf{e}_+\rangle\langle\mathbf{e}_+| + (-1)|\mathbf{e}_-\rangle\langle\mathbf{e}_-| = e^0|\mathbf{e}_+\rangle\langle\mathbf{e}_+| + e^{-i\pi}|\mathbf{e}_-\rangle\langle\mathbf{e}_-| \\ &= \exp(-i\pi|\mathbf{e}_-\rangle\langle\mathbf{e}_-|) = e^{-i\pi H_{\mathsf{h}}}, \qquad H_{\mathsf{h}} := |\mathbf{e}_-\rangle\langle\mathbf{e}_-|. \end{aligned}$$

In other words, $iH_{\mathsf{h}} \in \mathscr{A}(\mathcal{H})$ generates the Hadamard gate after evolving for time $t = \pi$, i.e., $U_t = \exp(-itH_{\mathsf{h}})$ solves the Schrödinger equation

$$\mathrm{d}U_t = -iH_{\mathsf{h}}U_t\,\mathrm{d}t, \qquad U_0 = \mathsf{I}_{\mathcal{H}}.$$

Notice that $H_{\mathsf{h}} \in \mathscr{P}(\mathcal{H})$ happens to be a rank-1 projection on the unit vector $\mathbf{e}_- \in \mathcal{H}$.

In the presence of noise, however, we instead have

$$\mathrm{d}U_t^{\upbeta} = -iH_{\mathsf{h}}U_t^{\upbeta} \circ \mathrm{d}\upbeta_t, \qquad U_0 = \mathsf{I}_{\mathcal{H}},$$

From Remark 1.4, we obtain the explicit form $U_t^{\upbeta} = \exp(-i\upbeta_t H_{\mathsf{h}})$.

Suppose we have a noisy pulse $\upbeta_t = t + \sqrt{\gamma}\upomega_t$. Then the fidelity between the desired pure state $|U_t\psi\rangle\langle U_t\psi|$ and the noisy pure state $|U_t^{\upbeta}\psi\rangle\langle U_t^{\upbeta}\psi|$ for any unit vector $\psi \in \mathcal{H}$ is

$$\mathsf{F}_t^{\upbeta}(\psi) := |\langle U_t^{\upbeta}\psi, U_t\psi\rangle|^2 = |\langle U_t^{\dagger}U_t^{\upbeta}\psi, \psi\rangle|^2 \quad \in [0,1].$$

Using the formulas obtained above, we easily deduce that

$$U_t^{\dagger}U_t^{\upbeta} = \exp\bigl(-i\sqrt{\gamma}\upomega_t H_{\mathsf{h}}\bigr) = |\mathbf{e}_+\rangle\langle\mathbf{e}_+| + e^{-i\sqrt{\gamma}\upomega_t}|\mathbf{e}_-\rangle\langle\mathbf{e}_-|.$$

Since $\{\mathbf{e}_+, \mathbf{e}_-\}$ forms an orthonormal basis for $\mathcal{H}$, $\psi = \psi_+\mathbf{e}_+ + \psi_-\mathbf{e}_-$, for which we obtain

$$\begin{aligned} \mathsf{F}_t^{\upbeta}(\psi) &= |\langle \psi_+\mathbf{e}_+ + \psi_- e^{-i\sqrt{\gamma}\upomega_t}\mathbf{e}_-, \psi_+\mathbf{e}_+ + \psi_-\mathbf{e}_-\rangle|^2 \\ &= ||\psi_+|^2 + |\psi_-|^2 e^{i\sqrt{\gamma}\upomega_t}|^2 = 1 - 2\bigl(1 - \cos(\sqrt{\gamma}\upomega_t)\bigr)|\psi_+|^2|\psi_-|^2. \end{aligned}$$

Since $|\psi_+|^2 + |\psi_-|^2 = 1$, we have that $|\psi_+|^2|\psi_-|^2 \le \frac{1}{2}$.

From this, we can deduce all statistical properties of the fidelity, e.g., its expectation

$$\mathbb{E}[\mathsf{F}_t^{\upbeta}(\psi)] = 1 - 2\bigl(1 - \mathbb{E}[\cos(\sqrt{\gamma}\upomega_t)]\bigr)|\psi_+|^2|\psi_-|^2 = 1 - 2\bigl(1 - e^{-\gamma t/2}\bigr)|\psi_+|^2|\psi_-|^2,$$

and variance

$$\begin{aligned} \mathbb{V}[\mathsf{F}_t^{\upbeta}(\psi)] &= \mathbb{E}\Bigl[(\mathsf{F}_t^{\upbeta}(\psi) - \mathbb{E}[\mathsf{F}_t^{\upbeta}(\psi)])^2\Bigr] = 4\mathbb{E}\Bigl[\bigl(\cos(\sqrt{\gamma}\upomega_t) - e^{-\gamma t/2}\bigr)^2\Bigr]|\psi_+|^4|\psi_-|^4 \\ &= 4\bigl(1 - 2\mathbb{E}[\cos(\sqrt{\gamma}\upomega_t)]e^{-\gamma t/2} + e^{-\gamma t}\bigr)^2|\psi_+|^4|\psi_-|^4 = 4\bigl(1 - e^{-\gamma t}\bigr)^2|\psi_+|^4|\psi_-|^4, \end{aligned}$$

where we used the fact that $\mathbb{E}[\cos(\sqrt{\gamma}\upomega_t)] = \exp(-\gamma t/2)$.

Exercise 1.1 Compute the fidelity of the Hadamard gate using the Lindblad equation.

Question: Can we improve the fidelity by implementing a different pulse?

In [3], we introduced a variational quantum algorithm to do just this, where the fidelity was used as a *regularizer* to a deterministic optimal control problem. For instance, given a desired pure state $|\psi_{\mathsf{d}}\rangle\langle\psi_{\mathsf{d}}| \in \mathscr{P}(\mathcal{H})$ at time $t = T$, one solves the problem

$$\min_{\mathsf{b}} \left\{ |\langle \psi_{\mathsf{d}}, U_T\, \mathbf{e}_0\rangle|^2 - \lambda \int_0^T \mathbb{E}[\mathsf{F}_t^{\beta}(\mathbf{e}_0)]\, \mathrm{d}t \right\},$$

subject to the deterministic Schrödinger equation

$$\mathrm{d}U_t = -iHU_t\, \mathrm{d}\mathsf{b}_t, \qquad U_0 = \mathsf{I}_{\mathcal{H}},$$

and where

$$\mathbb{E}[\mathsf{F}_t^{\beta}(\mathbf{e}_0)] = \tfrac{1}{2}\big(1 + \mathbb{E}[\cos(\mathsf{b}_t + \sqrt{\gamma}\,\omega_t)]\big).$$

Here, one is interested in determining a gate U_T that maps $\mathbf{e}_0$ to ψ_{d}, while keeping the fidelity expectation of the evolution under control.

2 Quantum Noise and Fock Space

As shown in the previous section, SSEs driven by classical noise provide a useful class of stochastic unravellings of Lindblad dynamics. However, this approach is fundamentally limited: it produces only Lindblad generators with *Hermitian* jump operators. To describe general irreversible quantum dynamics—including spontaneous emission, particle loss, and counting processes—we must move beyond classical noise and introduce *quantum noise*.

Recall that in Section 1, the stochastic dilation $\mathsf{a}_t = U_t\rho_0 U_t^\dagger$, led, after taking expectations, to a Lindblad equation with Hermitian noise operators $L_j^\dagger = L_j$. While such Lindblad operators model dephasing and diffusion-type noise, they cannot describe dissipative processes such as amplitude damping, where the jump operator is non-Hermitian. An important example is the jump operator $L = \sigma_- = |0\rangle\langle 1|$ on a single qubit describing *spontaneous emission* due to black body radiation.

From a physical perspective, this reflects the fact that classical noise models only randomize *phases* or *energies*. Truly quantum processes involve the exchange of quanta with an environment, and therefore require a non-commutative noise model.

Unitary dilations and infinite environments

A guiding principle in the theory of open quantum systems is that irreversibility arises from neglecting environmental degrees of freedom in the following sense. One considers the composite Hilbert space $\mathcal{H}_{\text{tot}} = \mathcal{H}_{\text{sys}}\otimes\mathcal{H}_{\text{env}}$, describing the *universe* and consists of the system $\mathcal{H}_{\text{sys}}$ and an environment $\mathcal{H}_{\text{env}}$. In this setting, one obtains a unitary evolution

$$U_t : \mathcal{H}_{\text{tot}} \to \mathcal{H}_{\text{tot}}$$

The reduced evolution of the system state is then obtained by tracing out the environment,

$$\rho_t = \operatorname{tr}_{\mathcal{H}_{\text{env}}}\left[U_t(\rho_0\otimes\rho_{\text{env}})U_t^\dagger\right] \in \mathscr{D}(\mathcal{H}_{\text{sys}}),$$

where $\rho_0 \in \mathscr{D}(\mathcal{H}_{\text{sys}})$ is the initial state on the system and $\rho_{\text{env}} \in \mathscr{D}(\mathcal{H}_{\text{env}})$ is a given state on the environment. Requiring *Markovianity* and time-homogeneity of the evolution inevitably forces the environment to possess infinitely many degrees of freedom. In continuous time, this naturally leads to a description in terms of bosonic quantum fields.

2.1 Bosonic Fock space

The *bosonic (or symmetric) Fock space* of a complex Hilbert space $\mathcal{K}$ is defined as

$$\mathfrak{F}(\mathcal{K}) := \bigoplus_{n\in\mathbb{N}_0} \mathcal{K}^{\odot n},$$

where $\mathcal{K}^{\odot n}$ denotes the n-fold symmetric tensor product, i.e.,

$$f \in \mathcal{K}^{\odot n} \quad\Leftrightarrow\quad f(\sigma(u_1,\dots,u_n)) = f(u_1,\dots,u_n) \quad \text{for any permutation } \sigma,$$

and by convention $\mathcal{K}^{\odot 0} := \mathbb{C}$. The bosonic Fock space $\mathfrak{F}(\mathcal{K})$ inherits the scalar product from $\mathcal{K}$ defined by

$$\langle \oplus u^{(n)}, \oplus v^{(n)}\rangle_{\mathfrak{F}(\mathcal{K})} := \sum_{n\in\mathbb{N}_0} \langle u^{(n)}, v^{(n)}\rangle_{\mathcal{K}^{\otimes n}}.$$

We define the *exponential vectors*

$$\mathsf{e}(u) = \bigoplus_{n\in\mathbb{N}_0} \frac{1}{\sqrt{n!}} u^{\otimes n}, \qquad u \in \mathcal{K},$$

and the distinguished vector $\Omega := \mathsf{e}(0) = 1 \oplus 0 \oplus \cdots \in \mathfrak{F}(\mathcal{K})$, called the *vacuum vector.*

It turns out that the set of exponential vectors

$$\mathfrak{E}(\mathcal{K}) := \big\{ e(u) : u \in \mathcal{K} \big\} \quad \text{is } \textit{total} \text{ in } \mathfrak{F}(\mathcal{K}),$$

i.e., its linear span is dense in $\mathfrak{F}(\mathcal{K})$. This fact will be helpful for us in the future. Since,

$$\langle \mathsf{e}(u), \mathsf{e}(v) \rangle_{\mathfrak{F}(\mathcal{K})} = \sum_{n\in\mathbb{N}_0} \frac{1}{n!} \langle u, v \rangle_{\mathcal{K}}^n = e^{\langle u,v\rangle_{\mathcal{K}}}, \qquad u, v \in \mathcal{K},$$

the exponential vectors are normalizable. These normalized exponential vectors

$$\psi(u) = e^{-\frac{1}{2}\|u\|_{\mathcal{K}}^2} \mathsf{e}(u), \qquad u \in \mathcal{K},$$

are called *coherent vectors.*

For time-continuous noise, the *canonical* choice is

$$\mathcal{K} = L^2(\mathbb{R}_+, \lambda; \mathbb{C}^d) \cong L^2(\mathbb{R}_+, \lambda) \otimes \mathbb{C}^d,$$

where $d \in \mathbb{N}$ represents the number of *noise channels* and λ is the Lebesgue measure. From now on, we will only consider this choice with $d = 1$ and call $\mathfrak{F}(\mathcal{K})$ our noise environment.

2.2 Creation, annihilation, and gauge processes

On the bosonic Fock space $\mathfrak{F}(\mathcal{K})$, one defines operator-valued processes on $\mathfrak{E}(\mathcal{K})$:

the *annihilation* processes $\mathsf{A}(t)$, the *creation* processes $\mathsf{A}^\dagger(t)$,
the *gauge (or counting)* processes $\mathsf{N}(t)$.

Heuristically, $\mathsf{A}(t)$ annihilates a quantum in channel j arriving before time $t \geq 0$, $\mathsf{A}^\dagger(t)$ creates such a quantum, and $\lambda(t)$ counts quanta in the channels. Together, these processes encode absorption, emission, and counting statistics in a unified operator-theoretic framework and form the building blocks of quantum stochastic calculus, serving as driving noises in the Hudson-Parthasarathy theory of quantum processes.

Definition 2.1 Let $\mathcal{D}$ be a total subset of a complex Hilbert space $\mathcal{H}$.

(i) A *random variable* χ is an element of $\mathscr{L}(\mathcal{D}; \mathcal{H})$, where

$$\mathscr{L}(\mathcal{D}; \mathcal{H}) := \Big\{ \mathsf{Z} : \mathcal{D} \to \mathcal{H} \ \text{linear} : \mathcal{D} \subset \operatorname{dom}(\mathsf{Z}) \cap \operatorname{dom}(\mathsf{Z}^\dagger) \Big\}.$$

(ii) A *stochastic process* in $\mathcal{H}$ is a family $(\chi(t))_{t\in\mathbb{R}_+}$ of random variables such that

$$\mathbb{R}_+ \ni t \mapsto \chi(t)\eta \quad \text{is Borel measurable for every } \eta \in \mathcal{D}.$$

Definition 2.2 A *martingale* is a map $\mathbb{R}_+ \ni t \mapsto \mathtt{m}_t \in \mathcal{K}$ satisfying

$$\mathtt{m}_t \in L^2([0,t], \lambda) \qquad \text{and} \qquad \mathbf{1}_{[0,s]} \mathtt{m}_t = \mathtt{m}_s \quad \text{for every } s \leq t.$$

Example 2.3 A simple martingale is $\mathtt{m}_t = \mathbf{1}_{[0,t]}$, which we shall call the *canonical* martingale. Other martingales include $\mathtt{m}_t = v\mathbf{1}_{[0,t]}$, where $v \in \mathcal{K}$ is an arbitrary function.

Notice that nothing about the definition above is stochastic in the usual sense.

Annihilation and creation processes. The annihilation and creation operators corresponding to a martingale $\mathtt{m}$ are defined on exponential vectors $\mathsf{e}(u)$, $u \in \mathcal{K}$ by

$$\mathsf{A}_{\mathtt{m}}(t)\mathsf{e}(u) := \langle \mathtt{m}_t, u\rangle \mathsf{e}(u), \qquad \mathsf{A}^\dagger_{\mathtt{m}}(t)\mathsf{e}(u) := \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\Big|_{\varepsilon=0} \mathsf{e}(u + \varepsilon \mathtt{m}_t).$$

These operators are densely defined, mutually adjoint, and they satisfy the so-called *canonical commutation relations*

$$[\mathsf{A}_{\mathtt{m}}(t), \mathsf{A}^\dagger_{\mathtt{m}}(s)] = \langle \mathtt{m}_t, \mathtt{m}_s\rangle\, \mathsf{I}_{\mathfrak{F}(\mathcal{K})}, \qquad [\mathsf{A}_{\mathtt{m}}(t), \mathsf{A}_{\mathtt{m}}(s)] = [\mathsf{A}^\dagger_{\mathtt{m}}(t), \mathsf{A}^\dagger_{\mathtt{m}}(s)] = 0. \tag{CCR}$$

Due to the martingale property of $\mathtt{m}$, we also have that

$$(\mathsf{A}_{\mathtt{m}}(t) - \mathsf{A}_{\mathtt{m}}(s))\mathsf{e}(u) = \langle \mathtt{m}_t - \mathtt{m}_s, u\rangle \mathsf{e}(u) = \langle \mathbf{1}_{[s,t]}\mathtt{m}_t, u\rangle \mathsf{e}(u).$$

These relations give rise to the quantum Itô table that we will see in the following section.

With respect to the coherent vector $\psi(u) \in \mathfrak{F}(\mathcal{K})$, $u \in \mathcal{K}$, one has

$$\langle \psi(u), \mathsf{A}_{\mathtt{m}}(t)\psi(u)\rangle = \langle \mathtt{m}_t, u\rangle = \overline{\langle \psi(u), \mathsf{A}^\dagger_{\mathtt{m}}(t)\psi(u)\rangle},$$
$$\langle \psi(u), [\mathsf{A}_{\mathtt{m}}(t), \mathsf{A}^\dagger_{\mathtt{m}}(s)]\psi(u)\rangle = \langle \mathtt{m}_t, \mathtt{m}_s\rangle.$$

In particular, in the vacuum vector $\Omega = \psi(0)$, we find

$$\langle \Omega, \mathsf{A}_{\mathtt{m}}(t)\Omega\rangle = \overline{\langle \Omega, \mathsf{A}^\dagger_{\mathtt{m}}(t)\Omega\rangle} = 0, \qquad \langle \Omega, \mathsf{A}_{\mathtt{m}}(t)\mathsf{A}^\dagger_{\mathtt{m}}(s)\Omega\rangle = \langle \mathtt{m}_t, \mathtt{m}_s\rangle,$$

which shows that the self-adjoint field operators $\mathsf{B}_{\mathtt{m}} = \mathsf{A}_{\mathtt{m}} + \mathsf{A}^\dagger_{\mathtt{m}}$ reproduce the covariance structure of classical Brownian motion for the canonical martingale $\mathtt{m}_t = \mathbf{1}_{[0,t]}$, i.e.,

$$\langle \Omega, \mathsf{B}_{\mathtt{m}}(t)\mathsf{B}^\dagger_{\mathtt{m}}(s)\Omega\rangle = t \wedge s.$$

Thus, classical noise is recovered as a commutative subtheory of quantum noise.

Exercise 2.1 Use the *Zassenhaus formula* and the fact that

$$[\mathsf{A}_{\mathtt{m}}(t), [\mathsf{A}_{\mathtt{m}}(t), \mathsf{A}^\dagger_{\mathtt{m}}(t)]] = [\mathsf{A}^\dagger_{\mathtt{m}}(t), [\mathsf{A}_{\mathtt{m}}(t), \mathsf{A}^\dagger_{\mathtt{m}}(t)]] = 0 \qquad \text{for all } t \geq 0,$$

to show that for every $r \in \mathbb{R}$,

$$\mathbb{E}_\Omega\big[e^{ir\mathsf{B}_{\mathtt{m}}(t)}\big] := \langle \Omega, e^{ir\mathsf{B}_{\mathtt{m}}(t)}\Omega\rangle = \exp\Big(-\tfrac{1}{2}r^2\|\mathtt{m}_t\|^2_{\mathcal{K}}\Big).$$

Conclude from this that, under the vacuum vector Ω, the field operator $\mathsf{B}(t)$ is a Gaussian *random variable* with mean 0 and variance $\Sigma = \|\mathtt{m}_t\|^2$.

What would change if we replace the vacuum Ω with a coherent vector $\psi(u)$?

Gauge processes. The gauge processes $\mathsf{N}(t)$ *counts* the number of quanta in the channel up to time $t \geq 0$. Their expectation on coherent vectors $\psi(u)$ is given by

$$\langle \psi(u), \mathsf{N}(t)\psi(u)\rangle = \int_{[0,t]} \overline{u}(s)u(s)\,\mathrm{d}s = \|u\|^2_{\mathcal{K}}.$$

In particular, its distribution is given by

$$\mathbb{E}_{\psi(u)}\big[e^{ir\mathsf{N}(t)}\big] := \langle \psi(u), e^{ir\mathsf{N}(t)}\psi(u)\rangle = \exp\Big((e^{ir} - 1)\|u\|^2_{\mathcal{K}}\Big),$$

i.e., in the coherent state $\psi(u)$, $\mathsf{N}(t)$ has a Poisson statistics with intensity $|u|^2\lambda$.

In these lectures, we only have time to focus on the creation and annihilation processes. A proper treatise of the gauge process requires more preparation, but its inclusion in the following theory may be done without too much trouble.

2.3 Fock-Wiener isometry

The Wiener-Itô-Segal isomorphism provides a precise mathematical link between classical and quantum noise, which identifies the canonical bosonic Fock space with an L^2-space over classical Wiener space.

More precisely, let $(\mathcal{C}(\mathbb{R}_+;\mathbb{R}), \mathcal{F}, \mathbb{W})$ be a classical Wiener probability space with canonical process $(X_t)_{t\in\mathbb{R}_+}$ with $X_t(\omega) = \omega(t)$, $\omega \in \mathcal{C}(\mathbb{R}_+;\mathbb{R})$. Then there exists a unitary isomorphism

$$\mathcal{U} : \mathfrak{F}(\mathcal{K}) \to L^2(\mathcal{C}(\mathbb{R}_+;\mathbb{R}), \mathbb{W}),$$

called the *Fock-Wiener (or Wiener-Itô-Segal) isometry*, with the following properties:

(i) The vacuum vector Ω is mapped to the constant function 1.

(ii) Exponential vectors correspond to stochastic exponentials of Brownian motion, i.e.,

$$\mathcal{U}\big(\psi(u\mathbf{1}_{[0,t]})\big)(\omega) = \exp\bigg(\int_0^t u(s)\,\mathrm{d}X_s(\omega) - \frac{1}{2}\|u\mathbf{1}_{[0,t]}\|_{\mathcal{K}}^2\bigg) \quad \text{for } t \geq 0.$$

Recall that the right-hand side is an exponential martingale and that the family of such functions is total in $L^2(\mathcal{C}(\mathbb{R}_+;\mathbb{R}), \mathbb{W})$.

(iii) Multiple Wiener integrals of order n correspond to the n-particle sector $\mathcal{K}^{\odot n}$.

Under this isometry, the self-adjoint field operator $\mathsf{B}(t)$ acts as multiplication by the classical Brownian motion W_t. In particular,

$$\mathcal{U}\mathsf{B}(t)\mathcal{U}^{-1} = W_t,$$

viewed as a multiplication operator on the commutative algebra $L^\infty(\mathcal{C}(\mathbb{R}_+;\mathbb{R}), \mathbb{W})$. This identification shows that classical stochastic calculus is faithfully embedded into quantum stochastic calculus as the restriction to a commuting subalgebra of field operators.

Remark 2.4 (1) Notice that the coherent states $\psi(u) \in \mathfrak{F}(\mathcal{K})$ play the role of changing the reference measure $\mathbb{W}$ by the drift field $u \in \mathcal{K}$.

(2) A similar construction holds for Poisson processes on the Skorokhod space. $\diamondsuit$

3 Hudson-Parthasarathy Theory

Having introduced quantum noise, we now describe the dynamics of systems driven by such noise. This is accomplished by the Hudson-Parthasarathy (HP) theory of quantum stochastic differential equations (QSDE), where the bosonic Fock space is used as a model for the environment Hilbert space, i.e., $\mathcal{H}_{\text{env}} = \mathfrak{F}(\mathcal{K})$. For simplicity, we consider a single noise channel, i.e., $d = 1$, and focus on the creation and annihilation processes. Henceforth, we consider an n-dimensional complex Hilbert space $\mathcal{H}_{\text{sys}}$ as our system and $\mathcal{H}_{\text{env}} = \mathfrak{F}(\mathcal{K})$ with $\mathcal{K} = L^2(\mathbb{R}_+)$ is our environment, which gives $\mathcal{H}_{\text{tot}} = \mathcal{H}_{\text{sys}} \otimes \mathcal{H}_{\text{env}}$.

Recall from Section 1 that the basic unitary evolution was derived by simply applying Itô's formula to $U_t = \exp(i\omega_t L)\mathsf{I}_{\mathcal{H}}$, where $L \in \mathscr{O}(\mathcal{H})$ is a Hermitian noise operator such that $i\omega_t L \in \mathscr{A}(\mathcal{H})$ is in the Lie algebra. One could think of doing something similar by considering

$$U_t = \exp(L \otimes \mathsf{A}_{\mathtt{m}}^{\dagger}(t) - L^{\dagger} \otimes \mathsf{A}_{\mathtt{m}}(t))\mathsf{I}_{\mathcal{H}_{\text{tot}}},$$

where $L \in \mathscr{B}(\mathcal{H}_{\text{sys}})$ and $\mathsf{A}_{\mathtt{m}}(t)$, $\mathsf{A}_{\mathtt{m}}^{\dagger}(t)$ are the field operators on $\mathcal{H}_{\text{env}}$. Notice that

$$\left(L \otimes \mathsf{A}_{\mathtt{m}}^{\dagger} - L^{\dagger} \otimes \mathsf{A}_{\mathtt{m}}\right)^{\dagger} = L^{\dagger} \otimes \mathsf{A}_{\mathtt{m}} - L \otimes \mathsf{A}_{\mathtt{m}}^{\dagger} = -\left(L \otimes \mathsf{A}_{\mathtt{m}}^{\dagger} - L^{\dagger} \otimes \mathsf{A}_{\mathtt{m}}\right),$$

i.e., $L \otimes \mathsf{A}_{\mathtt{m}}^{\dagger} - L^{\dagger} \otimes \mathsf{A}_{\mathtt{m}} \in \mathscr{A}(\mathcal{H}_{\text{tot}})$ is skew-Hermitian. Thus, U_t is a unitary evolution.

If we knew of an Itô formula for the creation and annihilation operators, then we would be able to derive a QSDE for the unitary process $U_t \in \mathscr{U}(\mathcal{H}_{\text{tot}})$. In general, one would also like to consider time-dependent system operators, i.e., $L = L(t)$, which will require us to introduce stochastic integrals of the form

$$\int_0^t L(s) \otimes \mathrm{d}\mathsf{A}_{\mathtt{m}}^{\dagger}(s), \quad \int_0^t L^{\dagger}(s) \otimes \mathrm{d}\mathsf{A}_{\mathtt{m}}(s).$$

To do so, we will need to understand the type of processes $L(t)$ we can integrate against. As in the classical Itô integral, we will require the notion of an *adapted process*.

3.1 Quantum Itô integrals and Itô calculus

Before we talk about quantum stochastic integrals, we need to introduce the quantum analog of a *filtration*. We begin by noting that for any $0 \leq t_1 < \cdots < t_n < +\infty$, we have the direct sum factorization

$$\mathcal{K} = L^2([0, t_1]) \oplus L^2([t_1, t_2]) \oplus \cdots \oplus L^2([t_n, +\infty)) =: \mathcal{K}_{t_1]} \oplus \mathcal{K}_{[t_1, t_2]} \oplus \cdots \oplus \mathcal{K}_{[t_n},$$

which gives the factorization $\mathfrak{F}(\mathcal{K}) = \mathfrak{F}(\mathcal{K}_{t_1]}) \otimes \mathfrak{F}(\mathcal{K}_{[t_1,t_2]}) \otimes \cdots \otimes \mathfrak{F}(\mathcal{K}_{[t_n})$. Denoting

$$\mathcal{F}_0 := \mathcal{H}_{\text{sys}}, \quad \mathcal{F}_{t]} := \mathcal{H}_{\text{sys}} \otimes \mathfrak{F}(\mathcal{K}_{t]}), \quad \mathcal{F}_{[s,t]} := \mathfrak{F}(\mathcal{K}_{[s,t]}), \quad \mathcal{F}_{[t} := \mathfrak{F}(\mathcal{K}_{[t}),$$

we then have the tensor product factorization of the time index

$$\mathcal{H}_{\text{tot}} = \mathcal{F}_{t_1]} \otimes \mathcal{F}_{[t_1,t_2]} \otimes \cdots \otimes \mathcal{F}_{[t_n},$$

which is associated with the factorization on the level of the vectors

$$\psi \otimes \mathsf{e}(u) = \psi \otimes \mathsf{e}(u\mathbf{1}_{[0,t_1]}) \otimes \mathsf{e}(u\mathbf{1}_{[t_1,t_2]}) \otimes \cdots \otimes \mathsf{e}(u\mathbf{1}_{[t_n,+\infty)})$$

The increasing family $(\mathcal{F}_{t]})_{t \geq 0}$ may then be used to define an adapted process. Roughly speaking, a process $(\chi(t))_{t \in \mathbb{R}_+}$ is adapted if for each $t \geq 0$, it leaves the space $\mathcal{F}_{[t}$ *ontouched*, i.e., it takes the form $\chi(t) \otimes \mathsf{I}_{[t}$ for every $t \geq 0$.

Definition 3.1 A stochastic process $(\chi(t))_{t\in\mathbb{R}_+}$ in $\mathcal{H}_{\text{tot}}$ is said to be $\mathcal{F}$-*adapted* to if the map $t\mapsto \chi(t)\psi\otimes \mathsf{e}(u)$ is measurable and there exists an operator $\mathsf{Z}(t)$ acting on $\mathcal{F}_{t]}$ such that

$$\chi(t)\psi\otimes \mathsf{e}(u) = \big(\mathsf{Z}(t)\psi\otimes \mathsf{e}(u\mathbf{1}_{[0,t]})\big) \otimes \mathsf{e}(u\mathbf{1}_{[t,+\infty)}),$$

for every $t\geq 0$, $\psi\in\mathcal{H}_{\text{sys}}$ and $u\in\mathcal{K}$.

Such a process is said to be *regular* if the map $t\mapsto \chi(t)\psi\otimes \mathsf{e}(u)$ is continuous.

Remark 3.2 (i) The processes $L\otimes \mathsf{A}^\dagger_{\mathtt{m}}(t)$, $L^\dagger\otimes \mathsf{A}_{\mathtt{m}}(t)$ are $\mathcal{F}$-adapted. Indeed,

$$\begin{aligned} L^\dagger\otimes \mathsf{A}_{\mathtt{m}}(t)\psi\otimes \mathsf{e}(u) &= (L^\dagger\psi)\otimes(\mathsf{A}_{\mathtt{m}}(t)\mathsf{e}(u)) \\ &= (L^\dagger\psi)\otimes(\langle \mathtt{m}_t, u\mathbf{1}_{[0,t]}\rangle \mathsf{e}(u\mathbf{1}_{[0,t]}))\otimes \mathsf{e}(u\mathbf{1}_{[t,+\infty)}). \end{aligned}$$

A similar result holds for $L\otimes \mathsf{A}^\dagger_{\mathtt{m}}(t)$.

(ii) For any $0\leq s<t<+\infty$, we have that

$$\begin{aligned} \big(\mathsf{A}_{\mathtt{m}}(t)-\mathsf{A}_{\mathtt{m}}(s)\big)\mathsf{e}(u) &= \big(\langle \mathtt{m}_t,u\rangle - \langle \mathtt{m}_s,u\rangle\big)\mathsf{e}(u) = \langle \mathtt{m}_t, u\mathbf{1}_{[s,t]}\rangle \mathsf{e}(u) \\ &= \mathsf{e}(u\mathbf{1}_{[0,s]})\otimes\big(\mathsf{A}_{\mathtt{m}}(t)-\mathsf{A}_{\mathtt{m}}(s)\big)\mathsf{e}(u\mathbf{1}_{[s,t]})\otimes \mathsf{e}(u\mathbf{1}_{[t,+\infty]}), \end{aligned}$$

i.e., $\mathsf{A}_{\mathtt{m}}(t)-\mathsf{A}_{\mathtt{m}}(s)$ acts only on the part $\mathcal{F}_{[s,t]}$, while leaving the rest untouched. This holds similarly for $\mathsf{A}^\dagger_{\mathtt{m}}(t)-\mathsf{A}^\dagger_{\mathtt{m}}(s)$. $\diamondsuit$

In the following, let M be either $\mathsf{A}_{\mathtt{m}}$ or $\mathsf{A}^\dagger_{\mathtt{m}}$ and for $0\leq t_1<\cdots<t_n<+\infty$, we consider *simple process* of the form

$$L := \sum_{j=0}^{n-1} L_j \mathbf{1}_{[t_j,t_{j+1})} + L_n\mathbf{1}_{[t_j,+\infty)}, \qquad L_j\in\mathscr{B}(\mathcal{F}_{t_j]}).$$

By construction, $(L(t))_{t\in\mathbb{R}_+}$ are $\mathcal{F}$-adapted processes in $\mathcal{H}_{\text{tot}}$. For these simple processes, we define the quantum Itô integral w.r.t. M as

$$\chi(t) := \int_0^t L(s)\,\mathrm{d}\mathsf{M}(s) := \sum_{j=0}^n L_j\big(\mathsf{M}(t\wedge t_{j+1})-\mathsf{M}(t\wedge t_j)\big),$$

with the convention $t_{n+1}=+\infty$.

Lemma 3.3. *The process* $\chi(t)$ *is regular and satisfies*

$$\langle \varphi\otimes \mathsf{e}(v), \chi(t)\psi\otimes \mathsf{e}(u)\rangle = \int_0^t \langle\varphi\otimes \mathsf{e}(v), L(s)\psi\otimes \mathsf{e}(u)\rangle \mu(\mathrm{d}s),$$

where

$$\mu = \begin{cases} u\,\overline{\mathtt{m}}_t\,\lambda & \text{if } \mathsf{M}=\mathsf{A}, \\ \overline{v}\,\mathtt{m}_t\,\lambda & \text{if } \mathsf{M}=\mathsf{A}^\dagger. \end{cases}$$

In particular, $\langle\varphi\otimes\Omega, \chi(t)\psi\otimes\Omega\rangle = 0$ *for every* $t\geq 0$ *and* $\psi,\varphi\in\mathcal{H}_{sys}$.

Lemma 3.4. *If* Υ *is another simple process with*

$$\Upsilon(t) := \int_0^t K(s)\,\mathrm{d}\mathsf{N}(s),$$

then

$$\langle \varphi\otimes\Omega, \chi(t)\Upsilon(t)\psi\otimes\Omega\rangle = \begin{cases} \int_0^t \langle \psi\otimes\Omega, LK\psi\otimes\Omega\rangle\, |\mathtt{m}_t|^2 \, \mathrm{d}\lambda & \textit{if } \mathsf{M} = \mathsf{A}_{\mathtt{m}} \textit{ and } \mathsf{N} = \mathsf{A}^\dagger_{\mathtt{m}}, \\ 0 & \textit{otherwise}, \end{cases}$$

and the quantum Itô product rule becomes

$$\mathrm{d}\big(\chi(t)\Upsilon(t)\big) = (\mathrm{d}\chi(t))\Upsilon(t) + \chi(t)\mathrm{d}\Upsilon(t) + L(t)K(t)\,\mathrm{d}\langle \mathsf{MN}\rangle(t),$$

with the following Itô table for $\mathrm{d}\langle\mathsf{MN}\rangle$*:*

$\mathrm{d}\mathsf{M}\backslash\mathrm{d}\mathsf{N}$	$\mathrm{d}\mathsf{A}$	$\mathrm{d}\mathsf{A}^\dagger$
$\mathrm{d}\mathsf{A}$	0	$\lvert\mathtt{m}_t\rvert^2\,\mathrm{d}\lambda$
$\mathrm{d}\mathsf{A}^\dagger$	0	0

The quantum Itô product rule in Lemma 3.4 allows us to further obtain an Itô formula for arbitrary smooth functions $f\colon \mathbb{C}\to\mathbb{C}$,

$$\mathrm{d}f(\chi(t)) = Df(\chi(t))\,\mathrm{d}\chi(t) + \tfrac{1}{2}D^2 f(\chi(t))\,\mathrm{d}\langle\chi^2\rangle(t).$$

We note that while most of the computations here have been performed for the vacuum state Ω, they can be carried out for arbitrary exponential vectors $\mathsf{e}(u)$.

As a special case, we find

$$\|\chi(t)\psi\otimes\Omega\|^2 = \begin{cases} \int_0^t \|L\psi\otimes\Omega\|^2\, |\mathtt{m}_t|^2\,\mathrm{d}\lambda & \text{if } \mathsf{M} = \mathsf{A}^\dagger_{\mathtt{m}}, \\ 0 & \text{otherwise}, \end{cases}$$

which will allow us to extend the stochastic integral to adapted processes that are not necessarily simple as in the classical case.

Definition 3.5 A $\mathcal{F}$-adapted process $(L(t))_{t\in\mathbb{R}_+}$ is said to be *stochastically integrable* if there exists a sequence $(L^n)_{n\in\mathbb{N}}$ of simple $\mathcal{F}$-adapted processes such that

$$\int_0^t \|(L^n - L)\psi\otimes\mathsf{e}(u)\|^2\,|v|^2\,\mathrm{d}\lambda \longrightarrow 0 \quad \text{for every } \psi\in\mathcal{H}_{\text{sys}},\ u,v\in\mathcal{K}.$$

We denote by $\mathbb{L}(\mathcal{F})$ the complex linear space of all stochastically integrable processes.

Proposition 3.6. *Let* $(L(t))_{t\in\mathbb{R}_+}$ *be any* $\mathcal{F}$*-adapted process satisfying*

(i) the map $t\mapsto L(t)\psi\otimes\mathsf{e}(u)$ *is continuous,*

(ii) $\sup_{s\in[0,t]}\|L(s)\psi\otimes\mathsf{e}(u)\| < +\infty$ *for every* $t\geq 0$*,*

for every $\psi\in\mathcal{H}_{\text{sys}}$ *and* $u\in\mathcal{K}$*. Then* $L\in\mathbb{L}(\mathcal{F})$*.*

3.2 Hudson-Parthasarathy equation

We now come to the main part of the course. Let $H \in \mathcal{O}(\mathcal{H}_{\mathrm{sys}})$, $L \in \mathcal{B}(\mathcal{H}_{\mathrm{sys}})$, and set

$$\chi(t) = -itH \otimes \mathsf{I}_{\mathcal{H}_{\mathrm{env}}} + L \otimes \mathsf{A}_{\mathtt{m}}^{\dagger}(t) - L^{\dagger} \otimes \mathsf{A}_{\mathtt{m}}(t) \quad \in \mathscr{A}(\mathcal{H}_{\mathrm{tot}}).$$

Then, from the Itô formula, one deduces that the unitary process $U_t \coloneqq \exp(\chi(t)) \in \mathscr{U}(\mathcal{H}_{\mathrm{tot}})$ satisfies the Hudson-Parthasarathy QSDE

$$\mathrm{d}U_t = \big(\mathrm{d}\chi(t) + \mathfrak{I}\, |\mathtt{m}_t|^2 \mathrm{d}t\big) U_t = \circ\, \mathrm{d}\chi(t)\, U_t, \qquad \mathfrak{I} = -\tfrac{1}{2}(L^{\dagger}L) \otimes \mathsf{I}_{\mathcal{H}_{\mathrm{env}}}, \tag{QSDE}$$

representing the joint evolution of system and environment, where we used the fact that

$$\mathrm{d}\langle \chi \rangle = -L^{\dagger}L \otimes \mathsf{I}_{\mathcal{H}_{\mathrm{env}}} |\mathtt{m}_t|^2 \mathrm{d}t.$$

Remark 3.7 Let $L = iK$ with $K \in \mathscr{O}(\mathcal{K}_{\mathrm{sys}})$. Then, $L^{\dagger} = -L$, and hence,

$$\chi(t) = -itH \otimes \mathsf{I}_{\mathcal{H}_{\mathrm{env}}} + iK \otimes \mathsf{B}_{\mathtt{m}}(t) \quad \in \mathscr{A}(\mathcal{H}_{\mathrm{tot}}),$$

i.e., we recover the scenario of Section 1 with $\mathsf{B}_{\mathtt{m}} = \mathsf{A}_{\mathtt{m}} + \mathsf{A}_{\mathtt{m}}^{\dagger}$ as introduced in Section 2.

Towards the Lindblad equation Recal from Section 2 that the density matrix is given by

$$\rho_t \coloneqq \mathrm{tr}_{\mathcal{H}_{\mathrm{env}}}\big[U_t(\rho_0 \otimes |\Omega\rangle\langle\Omega|) U_t^{\dagger}\big], \qquad \rho_0 \in \mathscr{D}(\mathcal{H}_{\mathrm{sys}}).$$

Let $\mathsf{b} \in \mathscr{O}(\mathcal{H}_{\mathrm{sys}})$ be an arbitrary observable. Then,

$$\langle \rho_t, \mathsf{b} \rangle_{\mathscr{O}(\mathcal{H}_{\mathrm{sys}})} = \mathrm{tr}_{\mathcal{H}_{\mathrm{tot}}}\big[\rho_0 \otimes |\Omega\rangle\langle\Omega| U_t^{\dagger}\, \mathsf{b} \otimes \mathsf{I}_{\mathcal{H}_{\mathrm{env}}} U_t\big].$$

Hence, we deduce the evolution for ρ_t by duality.

The following lemma will be essential in this endeavour. Its proof is straightforward with the use of the Itô table in Lemma 3.4.

Lemma 3.8. *Define* $j_t(\mathsf{b}) \coloneqq U_t^{\dagger}\, \mathsf{b} \otimes \mathsf{I}_{\mathcal{H}_{\mathrm{env}}} U_t$ *for any* $\mathsf{b} \in \mathscr{O}(\mathcal{H}_{sys})$. *Then,*

$$\mathrm{d}j_t(\mathsf{b}) = j_t\big(i[H, \mathsf{b}] + \mathscr{L}(\mathsf{b})|\mathtt{m}_t|^2\big)\, \mathrm{d}t - j_t\big([L, \mathsf{b}]\big) \mathsf{I}_{\mathcal{H}_{\mathrm{sys}}} \otimes \mathrm{d}\mathsf{A}_{\mathtt{m}}^{\dagger} - j_t\big([L^{\dagger}, \mathsf{b}]\big) \mathsf{I}_{\mathcal{H}_{\mathrm{sys}}} \otimes \mathrm{d}\mathsf{A}_{\mathtt{m}}.$$

Due to Lemma 3.3 and Remark 1.2, we then deduce

$$\begin{aligned}
\mathrm{d}\langle \rho_t, \mathsf{b} \rangle_{\mathscr{O}(\mathcal{H}_{\mathrm{sys}})} &= \mathrm{tr}_{\mathcal{H}_{\mathrm{tot}}}\big[\rho_0 \otimes |\Omega\rangle\langle\Omega| U_t^{\dagger}\big(i[H, \mathsf{b}] + \mathscr{L}(\mathsf{b})|\mathtt{m}_t|^2\big) \otimes \mathsf{I}_{\mathcal{H}_{\mathrm{env}}} U_t\big] \mathrm{d}t \\
&= \mathrm{tr}_{\mathcal{H}_{\mathrm{sys}}}\big[\rho_t\big(i[H, \mathsf{b}] + \mathscr{L}(\mathsf{b})\big)|\mathtt{m}_t|^2\big] \mathrm{d}t \\
&= \mathrm{tr}_{\mathcal{H}_{\mathrm{sys}}}\big[\big(-i[H, \rho_t] + \mathscr{L}^*(\rho_t)|\mathtt{m}_t|^2\big)\mathsf{b}\big] \mathrm{d}t \\
&= \langle -i[H, \rho_t] + \mathscr{L}^*(\rho_t)|\mathtt{m}_t|^2, \mathsf{b} \rangle_{\mathscr{O}(\mathcal{H}_{\mathrm{sys}})}\, \mathrm{d}t,
\end{aligned}$$

which yields the Lindblad equation (GKSL) by duality.

3.3 Example: 1-qubit fidelity under amplitude damping

No driving Hamiltonian

As in Section 1.4, we consider the qubit model with $\mathcal{H}_{\mathrm{sys}} = \mathbb{C}^2$. In the absence of a coherent driving Hamiltonian $H \in \mathscr{O}(\mathcal{H}_{\mathrm{sys}})$ and noise operator

$$L = \sigma_- = |\mathbf{e}_0\rangle\langle\mathbf{e}_1|, \qquad L^\dagger =: \sigma_+ = |\mathbf{e}_1\rangle\langle\mathbf{e}_0| \in \mathscr{B}(\mathcal{H}_{\mathrm{sys}}),$$

the QSDE with a general martingale $\mathtt{m}_t$ becomes

$$\mathrm{d}U_t = \Big(\sigma_-\otimes \mathrm{dA}^\dagger_{\mathtt{m}}(t) - \sigma_+\otimes \mathrm{dA}_{\mathtt{m}}(t) - \tfrac{1}{2}\sigma_+\sigma_-\otimes\mathsf{I}_{\mathcal{H}_{\mathrm{env}}}|\mathtt{m}_t|^2\mathrm{d}t\Big)U_t, \tag{3.1}$$

with the explicit solution $U_t = \exp(\chi(t)) \in \mathscr{U}(\mathcal{H}_{\mathrm{tot}})$, where

$$\chi(t) = \sigma_-\otimes\mathsf{A}^\dagger_{\mathtt{m}}(t) - \sigma_+\otimes\mathsf{A}_{\mathtt{m}}(t) \quad \in \mathscr{A}(\mathcal{H}_{\mathrm{tot}}).$$

We are interested in the fidelity of the pure state $\rho := |\psi\rangle\langle\psi| \in \mathscr{P}(\mathcal{H}_{\mathrm{sys}})$ under noise. Since ρ is pure, the (random) fidelity between ρ and its noisy counterpart is given by

$$\mathsf{F}_t(\psi) = \mathrm{tr}_{\mathcal{H}_{\mathrm{sys}}}\big[\rho\otimes\mathsf{I}_{\mathcal{H}_{\mathrm{env}}}U_t\,\rho\otimes|\Omega\rangle\langle\Omega|U_t^\dagger\big],$$

which is an operator on the environment $\mathcal{H}_{\mathrm{ent}}$, i.e., it is a Fock-space operator. To determine its distribution in the vacuum vector Ω, one would determine its characteristic function $\mathbb{E}_\Omega[e^{ir\mathsf{F}_t(\psi)}]$, which may be rather tedious—I have yet to discover a convenient way of doing this.

Its expectation in the vacuum vector Ω, however, can be obtained from

$$\begin{aligned}\mathbb{E}_\Omega[\mathsf{F}_t(\psi)] &= \mathrm{tr}_{\mathcal{H}_{\mathrm{env}}}\big[|\Omega\rangle\langle\Omega|\,\mathrm{tr}_{\mathcal{H}_{\mathrm{sys}}}\big[\rho\otimes\mathsf{I}_{\mathcal{H}_{\mathrm{env}}}U_t\,\rho\otimes|\Omega\rangle\langle\Omega|U_t^\dagger\big]\big]\\ &= \mathrm{tr}_{\mathcal{H}_{\mathrm{tot}}}\big[|\psi\otimes\Omega\rangle\langle\psi\otimes\Omega, U_t\,\psi\otimes\Omega\rangle\langle\psi\otimes\Omega|U_t^\dagger\big] = |\langle\psi\otimes\Omega, U_t\,\psi\otimes\Omega\rangle|^2.\end{aligned}$$

We can make use of Lemma 3.3 to obtain

$$\begin{aligned}\mathrm{d}\langle\mathbf{e}_0\otimes\Omega, U_t\,\mathbf{e}_j\otimes\Omega\rangle &= \langle\mathbf{e}_1\otimes\Omega, U_t\,\mathrm{dA}^\dagger_{\mathtt{m}}(t)\,\mathbf{e}_j\otimes\Omega\rangle = 0\\ \mathrm{d}\langle\mathbf{e}_1\otimes\Omega, U_t\,\mathbf{e}_j\otimes\Omega\rangle &= -\langle\mathbf{e}_0\otimes\Omega, U_t\,\mathrm{dA}_{\mathtt{m}}(t)\,\mathbf{e}_j\otimes\Omega\rangle - \tfrac{1}{2}\langle\mathbf{e}_1\otimes\Omega, U_t\,\mathbf{e}_j\otimes\Omega\rangle|\mathtt{m}_t|^2\mathrm{d}t\\ &= -\tfrac{1}{2}\langle\mathbf{e}_1\otimes\Omega, U_t\,\mathbf{e}_j\otimes\Omega\rangle|\mathtt{m}_t|^2\mathrm{d}t, \qquad j\in\{0,1\},\end{aligned}$$

which then implies

$$\langle\mathbf{e}_i\otimes\Omega, U_t\,\mathbf{e}_j\otimes\Omega\rangle = \begin{cases}1 & \text{for } i=j=0,\\ 0 & \text{for } i\neq j,\\ e^{-\frac{1}{2}\|\mathtt{m}_t\|^2_{\mathcal{K}}} & \text{for } i=j=1.\end{cases}$$

Since $\psi = \psi_0\mathbf{e}_0 + \psi_1\mathbf{e}_1$, we have that

$$\langle\psi\otimes\Omega, U_t\,\psi\otimes\Omega\rangle = |\psi_0|^2 + |\psi_1|^2 e^{-\frac{1}{2}\|\mathtt{m}_t\|^2_{\mathcal{K}}},$$

from which we can obtain an explicit formula

$$\mathbb{E}_\Omega[\mathsf{F}_t(\psi)] = \Big(|\psi_0|^2 + |\psi_1|^2 e^{-\frac{1}{2}\|\mathtt{m}_t\|^2_{\mathcal{K}}}\Big),$$

indicating no influence of noise in the ground state $|\mathbf{e}_0\rangle\langle\mathbf{e}_0|$ and an exponential decay for the state $|\mathbf{e}_1\rangle\langle\mathbf{e}_1|$ due to noise with a rate given by the martingale $\mathtt{m}_t$.

Exercise 3.1 Determine the characteristic function of the Fock-space operator $\mathrm{tr}_{\mathcal{H}_{\mathrm{sys}}}[|\mathbf{e}_1\rangle\langle\mathbf{e}_1|U_t]$.

With Detuning Hamiltonian

We now consider the driving Hamiltonian $H_{\mathsf{d}} := |\mathbf{e}_1\rangle\langle\mathbf{e}_1| \in \mathscr{O}(\mathcal{H}_{\mathrm{sys}})$, called the *detuning* Hamiltonian, which turns out to be equal to $L^\dagger L = \sigma_+\sigma_-$. A naive question one could ask is whether the noise's effect could be mitigated by this Hamiltonian.

Assuming the canonical martingale $\mathtt{m}_t = \mathbf{1}_{[0,t]}$, the QSDE becomes

$$\mathrm{d}U_t = \big(-iH_{\mathsf{d}}\otimes\mathsf{I}_{\mathcal{H}_{\mathrm{env}}}\,\mathrm{d}\mathsf{b}(t) + \sqrt{\gamma}\sigma_-\otimes\mathrm{d}\mathsf{A}^\dagger(t) - \sqrt{\gamma}\sigma_+\otimes\mathrm{d}\mathsf{A}(t) - \tfrac{\gamma}{2}\sigma_+\sigma_-\otimes\mathsf{I}_{\mathcal{H}_{\mathrm{env}}}\,\mathrm{d}t\big)U_t^\alpha,$$

with the explicit solution $U_t = \exp(\Upsilon(t))$, where

$$\Upsilon(t) = -iH_{\mathsf{d}}\otimes\mathsf{I}_{\mathcal{H}_{\mathrm{env}}}\mathsf{b}(t) + \sqrt{\gamma}\sigma_-\otimes\mathsf{A}^\dagger(t) - \sqrt{\gamma}\sigma_+\otimes\mathsf{A}(t) \quad \in \mathscr{A}(\mathcal{H}_{\mathrm{tot}}).$$

Computing the fidelity expectation now seems more complicated.

Let us consider the operator $V_t := e^{i\mathsf{b}(t)H_{\mathsf{d}}}\otimes\mathsf{I}_{\mathcal{H}_{\mathrm{env}}}U_t$ and its corresponding evolution

$$\begin{aligned}
\mathrm{d}V_t &= iH_{\mathsf{d}}\otimes\mathsf{I}_{\mathcal{H}_{\mathrm{env}}}V_t\,\mathrm{d}\mathsf{b}(t) + e^{i\mathsf{b}(t)H_{\mathsf{d}}}\otimes\mathsf{I}_{\mathcal{H}_{\mathrm{env}}}\mathrm{d}U_t\\
&= e^{i\mathsf{b}(t)H_{\mathsf{d}}}\otimes\mathsf{I}_{\mathcal{H}_{\mathrm{env}}}\big(\sigma_-\otimes\mathrm{d}\mathsf{A}^\dagger(t) - \sigma_+\otimes\mathrm{d}\mathsf{A}(t) - \sigma_+\sigma_-\otimes\mathsf{I}_{\mathcal{H}_{\mathrm{env}}}\,\mathrm{d}t\big)U_t\\
&= \big(\sigma_-^{\mathsf{b}}(t)\otimes\mathrm{d}\mathsf{A}^\dagger(t) - \sigma_+^{\mathsf{b}}(t)\otimes\mathrm{d}\mathsf{A}(t) - \sigma_+^{\mathsf{b}}(t)\sigma_-^{\mathsf{b}}(t)\otimes\mathsf{I}_{\mathcal{H}_{\mathrm{env}}}\,\mathrm{d}t\big)V_t,
\end{aligned}$$

with $\sigma_\pm^{\mathsf{b}}(t) = e^{i\mathsf{b}(t)H_{\mathsf{d}}}\sigma_\pm e^{-i\mathsf{b}(t)H_{\mathsf{d}}}$. Next, we observe that $\sigma_\pm^{\mathsf{b}}$ solves the initial value problem

$$\mathrm{d}\sigma_\pm^{\mathsf{b}}(t) = [H_{\mathsf{d}}, \sigma_\pm^{\mathsf{b}}(t)]\,\mathrm{d}\mathsf{b}(t), \qquad \sigma_\pm^{\mathsf{b}}(0) = \sigma_\pm,$$

with the unique solution $\sigma_\pm^H(t) = e^{i\mathsf{b}(t)[H_{\mathsf{d}},\cdot]}\sigma_\pm$. Since $[H_{\mathsf{d}}, \sigma_\pm] = \pm\sigma_\pm$, which implies that $\sigma_\pm$ are eigenvectors of the operator $[H_{\mathsf{d}}, \cdot]$ with eigenvalues ± 1, respectively, we then deduce that

$$\sigma_\pm^{\mathsf{b}}(t) = e^{i\mathsf{b}(t)[H_{\mathsf{d}},\cdot]}\sigma_\pm = e^{\pm i\mathsf{b}(t)}\sigma_\pm.$$

Consequently, the evolution for V_t now reads

$$dV_t = \big(e^{-i\mathsf{b}(t)}\sigma_-\otimes\mathrm{d}\mathsf{A}^\dagger(t) - e^{i\mathsf{b}(t)}\sigma_+\otimes\mathrm{d}\mathsf{A}(t) - \sigma_+\sigma_-\otimes\mathsf{I}_{\mathcal{H}_{\mathrm{env}}}\,\mathrm{d}t\big)V_t.$$

Finally, we claim that the evolution above may be reduced to an evolution of the form (3.1) for an appropriate choice of martingale $\mathtt{m}_t$. To see this, we set $\mathtt{m}_t(s) := e^{-i\mathsf{b}(s)}\mathbf{1}_{[0,t]}(s)$. Then,

$$\mathsf{A}_{\mathtt{m}}(t)\mathbf{e}(u) = \langle\mathtt{m}_t, u\rangle\mathbf{e}(u) = \left(\int_0^t e^{i\mathsf{b}(s)}\,u(s)\,\mathrm{d}s\right)\mathbf{e}(u),$$

for every exponential vector $\mathbf{e}(u)$, and therefore,

$$\mathrm{d}\mathsf{A}_{\mathtt{m}}(t)\mathbf{e}(u) = e^{i\mathsf{b}(t)}\,u(t)\mathbf{e}(u) = e^{i\mathsf{b}(t)}\,\mathrm{d}\mathsf{A}(t)\mathbf{e}(u).$$

By the totality of the exponential vectors, we conclude that the evolution of V_t may be equivalently expressed in the form (3.1) with martingale $\mathtt{m}_t(s) := e^{-i\mathsf{b}(s)}\mathbf{1}_{[0,t]}(s)$.

Based on the results above for the case with no driving Hamiltonian, we see that adding a detuning Hamiltonian cannot mitigate the noise. Indeed, for the case $\psi = \mathbf{e}_1$, we find that

$$\begin{aligned}
\langle\psi\otimes\Omega, U_t\,\psi\otimes\Omega\rangle &= \langle e^{i\mathsf{b}(t)H_{\mathsf{d}}}\otimes\mathsf{I}_{\mathcal{H}_{\mathrm{env}}}\psi\otimes\Omega, V_t\,\psi\otimes\Omega\rangle\\
&= e^{-i\mathsf{b}(t)}\langle\mathbf{e}_1\otimes\Omega, V_t\,\mathbf{e}_1\otimes\Omega\rangle = e^{-i\mathsf{b}(t)}e^{-\frac{1}{2}\|\mathtt{m}_t\|_{\mathcal{K}}^2},
\end{aligned}$$

and hence, $\mathbb{E}_\Omega[\mathsf{F}_t(\psi)] = e^{-\frac{1}{2}t}$, which is the same for the case without the detuning Hamiltonian.