Quantum Stochastic Processes for Quantum Computing

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Abstract. These lectures introduce stochastic models for noisy quantum dynamics with applications to quantum computing. We review stochastic Schrödinger equations and Lindblad dynamics as effective descriptions of open quantum systems, and illustrate their use in analyzing single-qubit gate fidelity. To model genuinely quantum noise, we introduce bosonic Fock space and develop the Hudson–Parthasarathy theory of quantum stochastic differential equations, providing a unitary dilation of Markovian open-system evolution with general jump operators. The theory is illustrated through explicit qubit examples, including dephasing and amplitude damping, with an emphasis on conceptual clarity.

Contents

References

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1 Classical noncommutative processes

As we have seen, a quantum circuit is essentially a noisy, controlled Schrödinger evolution. From a physical perspective, this noise arises from the unavoidable interaction between the system of interest and its environment. As a result, realistic quantum devices cannot be modeled as closed systems evolving unitarily on a Hilbert space, but rather as *open quantum systems*, whose effective dynamics are irreversible.

Throughout these notes, we denote by $\mathcal{H}$ a complex Hilbert space and $\mathscr{B}(\mathcal{H})$ the space of bounded operators on $\mathcal{H}$. We further consider the following subspaces of $\mathscr{B}(\mathcal{H})$ that play a distinguishing role:

$$\begin{aligned}
\mathscr{U}(\mathcal{H}) &:= \{\mathsf{a} \in \mathscr{B}(\mathcal{H}) : \mathsf{a}^\dagger \mathsf{a} = \mathsf{I}_{\mathcal{H}}\} && \textit{(unitary operators)}\\
\mathscr{O}(\mathcal{H}) &:= \{\mathsf{a} \in \mathscr{B}(\mathcal{H}) : \mathsf{a}^\dagger = \mathsf{a}\} && \textit{(observables)}\\
\mathscr{D}(\mathcal{H}) &:= \{\mathsf{a} \in \mathscr{O}(\mathcal{H}) : \mathsf{a} \succeq 0,\ \mathrm{tr}[\mathsf{a}] = 1\} && \textit{(density operators)}\\
\mathscr{P}(\mathcal{H}) &:= \{\mathsf{a} \in \mathscr{O}(\mathcal{H}) : \mathsf{a} = |\psi\rangle\langle\psi|,\ \|\psi\|_{\mathcal{H}} = 1\} && \textit{(pure states)}
\end{aligned}$$

where $\mathsf{a}^\dagger := \bar{\mathsf{a}}^\top$ is the hermitian conjugate of an operator $\mathsf{a} \in \mathscr{B}(\mathcal{H})$ and we will adopt the *bra-ket* notation to distinguish between primal vectors $|\psi\rangle$ (*ket*) and dual vectors $\langle\psi|$ (*bra*) for any vector $\psi \in \mathcal{H}$. Notice that $\mathscr{P}(\mathcal{H})$ is simply the subset of rank-1 projection operators.

1.1 Lindblad equation

A wide range of mathematical and physical models have been developed to describe open quantum systems, spanning microscopic Hamiltonian descriptions of system-environment interactions to phenomenological effective equations. Among these, one of the simplest and most widely used frameworks is provided by the *Lindblad (or Gorini-Kossakowski-Sudarshan-Lindblad, GKSL) equation.* Its importance stems from the fact that it provides the most general form of a Markovian, time-homogeneous quantum evolution compatible with the basic principles of quantum mechanics.

Rather than describing the system by a wave function, the Lindblad equation governs the evolution of the density operator $t \mapsto \rho_t \in \mathscr{D}(\mathcal{H})$, which encodes both classical and quantum uncertainty. The evolution with initial datum $\rho_0 \in \mathscr{D}(\mathcal{H})$ is given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho_t = -i[H, \rho_t] + \mathscr{L}^*(\rho_t), \qquad \mathscr{L}^*(\rho) := \tfrac{1}{2}\sum\nolimits_j \big(2L_j\rho L_j^\dagger - L_j^\dagger L_j\rho - \rho L_j^\dagger L_j\big), \tag{GKSL}$$

where $H \in \mathscr{O}(\mathcal{H})$ is a given system Hamiltonian, $\mathscr{L}^*$ is the *Lindblad (super)-operator*, and $L_j \in \mathscr{B}(\mathcal{H})$ are mutually orthogonal *jump operators*, describing the coupling to the environment. The first term represents coherent unitary evolution, while the second term captures irreversible processes such as decoherence, relaxation, and dissipation.

Remark 1.1 If $L_j \in \mathscr{O}(\mathcal{H})$ are observables, then the Lindblad operator takes the form

$$\mathscr{L}^*(\rho) = -\tfrac{1}{2}\sum\nolimits_j [L_j, [L_j, \rho]].$$

Remark 1.2 The use of * in $\mathscr{L}^*$ indicates that $\mathscr{L}^*$ is the adjoint of an operator $\mathscr{L}$ under an appropriate scalar product. In this case, the scalar product considered is the Hilbert-Schmidt scalar product $\langle \mathsf{a}, \mathsf{b}\rangle_{\mathscr{O}(\mathcal{H})} := \mathrm{tr}[\mathsf{a}^\dagger \mathsf{b}]$ since the objects in question are observables. Therefore,

$$\langle \mathscr{L}^*(\mathsf{a}), \mathsf{b}\rangle_{\mathscr{O}(\mathcal{H})} = \mathrm{tr}[(\mathscr{L}^*(\mathsf{a}))^\dagger \mathsf{b}] = \mathrm{tr}[\mathscr{L}^*(\mathsf{a})\mathsf{b}] = \mathrm{tr}[\mathsf{a}\mathscr{L}(\mathsf{b})] = \langle \mathsf{a}, \mathscr{L}(\mathsf{b})\rangle_{\mathscr{O}(\mathcal{H})},$$

with

$$\mathscr{L}(\mathsf{a}) = \tfrac{1}{2}\sum\nolimits_j \big(2L_j^\dagger \mathsf{a} L_j - L_j^\dagger L_j \mathsf{a} - \mathsf{a} L_j^\dagger L_j\big).$$

A thorough informal derivation of the Lindblad equation is lengthy and complex, so we will skip it here. However, from an operational point of view, the density operator ρ is an ensemble of pure states $\mathsf{a} = |\psi\rangle\langle\psi| \in \mathscr{P}(\mathcal{H})$. In other words, ρ can be interpreted as the expectation of the canonical random variable under a probability measure P on pure states $\mathscr{P}(\mathcal{H})$, i.e.,

$$\rho = \int_{\mathscr{P}(\mathcal{H})} \mathsf{a}\,\mathsf{P}(\mathrm{d}\mathsf{a}) \ \in \mathscr{D}(\mathcal{H}).$$

In the following, we will be interested in *stochastic dilations* or *stochastic unravellings* of the Lindblad equation, i.e., we will look at $\mathscr{P}(\mathcal{H})$-valued processes for which their expectation solves the Lindblad equation. The following sections provide examples of such processes, where information about the environment is embedded into the model.

1.2 Stochastic Schrödinger equation

In the context of Rydberg atoms, the optical control system is a primary source of noise. In the semiclassical limit of light-matter interactions, such noise sources can be considered classical. Without going into the details of the physics, we introduce the stochastic Schrödinger equation, which describes the evolution of a quantum system driven by classical noise sources.

Given an observable $L \in \mathscr{O}(\mathcal{H})$ and a real-valued smooth process β_t, the Schrödinger equation driven by the time-dependent observable $\dot{\beta}_t L \in \mathscr{O}(\mathcal{H})$ may be expressed as an evolution in the space of unitaries $\mathscr{U}(\mathcal{H})$:

$$\mathrm{d}U_t = iLU_t\,\mathrm{d}\beta_t, \qquad U_0 = \mathsf{I}_{\mathcal{H}}.$$

In this simple case, the solution may be explicitly expressed as

$$U_t = \exp\big(i(\beta_t - \beta_0)L\big)\mathsf{I}_{\mathcal{H}}.$$

Remark 1.3 When $\mathcal{H} = \mathbb{C}^n$, $\mathscr{U}(\mathcal{H})$ is a compact Lie group with Lie algebra

$$\mathscr{A}(\mathcal{H}) := \big\{\mathsf{a} \in \mathscr{B}(\mathcal{H}) : \mathsf{a}^\dagger = -\mathsf{a}\big\} = \big\{i\mathsf{a} \in \mathscr{B}(\mathcal{H}) : \mathsf{a} \in \mathscr{O}(\mathcal{H})\big\},$$

i.e., the space of skew-hermitian matrices. The Lie algebra $\mathscr{A}(\mathcal{H})$ can be equipped with a real inner product given by the Hilbert-Schmidt scalar product

$$\langle \mathsf{a}, \mathsf{b}\rangle_{\mathscr{A}(\mathcal{H})} := -\operatorname{tr}(\mathsf{a}^\dagger \mathsf{b}) = \Re \operatorname{tr}(\mathsf{a}\mathsf{b}^\dagger),$$

which is positive-definite on $\mathscr{A}(\mathcal{H})$. The associated norm is then

$$|\mathsf{a}|^2_{\mathscr{A}(n)} = \langle \mathsf{a}, \mathsf{a}\rangle_{\mathscr{A}(n)} = \operatorname{tr}(\mathsf{a}\mathsf{a}^\dagger).$$

Similarly, the space $\mathscr{O}(\mathcal{H})$ of observables may be equipped with the Hilbert-Schmidt scalar product $\langle \mathsf{a}, \mathsf{b}\rangle_{\mathscr{O}(\mathcal{H})} = \operatorname{tr}(\mathsf{a}^\dagger \mathsf{b}) \in \mathbb{R}$.

From a geometrical perspective, the unitary evolution is the exponential map applied to the time-dependent right-invariant vector field $\mathfrak{a}_t : \mathscr{U}(\mathcal{H}) \to \mathscr{A}(\mathcal{H})$, $\mathfrak{a}_t(U) = i\dot{\beta}_t HU = \mathfrak{a}_t(\mathsf{I}_{\mathcal{H}})U$. Throughout, we will consider right-multiplication, with U always acting on the right. ♢

In the presence of noise, however, β_t may no longer be smooth. Nevertheless, if $\beta_t = \omega_t$ is the Brownian motion, then the Itô formula applies and we get for $U_t := \exp(i\omega_t L)\mathsf{I}_{\mathcal{H}}$,

$$\mathrm{d}U_t = \big(iL\,\mathrm{d}\omega_t - \tfrac{1}{2}L^2\,\mathrm{d}t\big)U_t = iLU_t \circ \mathrm{d}\omega_t,$$

where $\circ\,\mathrm{d}\omega_t$ denotes the Stratonovich integral. This is precisely a stochastic Schrödinger equation with one noise operator L and one driving noise ω_t.

1.3 Unitary-valued processes

More generally, we consider a set $\mathscr{L} = \{H, L_1, \dots, L_d\} \subset \mathscr{O}(\mathcal{H})$ of orthonormal observables on the n-dimensional complex Hilbert space $\mathcal{H}$ under the Hilbert-Schmidt scalar product on $\mathscr{O}(\mathcal{H})$, where $1 \le d \le n^2$ is the number of noise channels. These observables will often be called noise operators. Further, let $\omega_t^1, \dots, \omega_t^d$ be independent standard real-valued Brownian motions and $\beta_t^1, \dots, \beta_t^d$ be Itô processes of the form

$$\beta_t^j = \beta_0^j + \mathsf{b}_t^j + \sqrt{\gamma_j}\,\omega_t^j, \qquad j = 1, \dots, d,$$

where b_t^j is an absolutely continuous process and $\gamma_j > 0$.

Define the $\mathscr{O}(\mathcal{H})$–valued (possibly degenerate) Brownian driver

$$\chi_t = -itH + \sum\nolimits_j iL_j\,\beta_t^j \quad \in \mathscr{A}(\mathcal{H}).$$

The intrinsic $\mathscr{U}(\mathcal{H})$-valued diffusion process solves the Stratonovich SDE

$$\mathrm{d}U_t = \circ\,\mathrm{d}\chi_t\,U_t, \qquad U_0 = \mathsf{I}_{\mathcal{H}}, \tag{SSE}$$

or in components

$$\mathrm{d}U_t = -iHU_t\,\mathrm{d}t + \sum\nolimits_j iL_jU_t\,\mathrm{d}\mathsf{b}_t^j + \sum\nolimits_j \sqrt{\gamma_j}\,iL_jU_t \circ \mathrm{d}\omega_t^j,$$

where H is a system Hamiltonian. The matrix-valued quadratic variation of χ is

$$\mathrm{d}\langle\chi\rangle_t = \sum\nolimits_j \gamma_j\,iL_j \otimes iL_j\,\mathrm{d}t.$$

If $d = n^2$, the driver is *elliptic* (non-degenerate). If $d < n^2$, the covariance has rank d and the process is *hypoelliptic* (degenerate), exploring only the connected subgroup

$$\exp(\mathscr{A}_{\mathscr{L}}) \subset \mathscr{U}(\mathcal{H}), \quad \mathscr{A}_{\mathscr{L}} = \mathrm{Lie}\{iH, iL_1, \dots, iL_r\} \subset \mathscr{A}(\mathcal{H}).$$

Clearly, there are situations where $\mathscr{A}_{\mathscr{L}} = \mathscr{A}(\mathcal{H})$ for $d < n^2$, in which case, the full group of unitaries is explored, i.e., $\exp(\mathscr{A}_{\mathscr{L}}) = \mathscr{U}(\mathcal{H})$.

In Itô form, the equivalent SDE reads

$$\mathrm{d}U_t = \big(\mathrm{d}\chi_t + \mathfrak{I}\,\mathrm{d}t\big)U_t, \qquad \mathfrak{I} := -\tfrac{1}{2}\sum\nolimits_j \gamma_j L_j^2,$$

where $\mathfrak{I}$ is the Laplace-Beltrami operator associated with the right-invariant connection.

Remark 1.4 When $H = 0$, $d = 1$, we find, as in the deterministic case, the explicit solution

$$U_t = \exp\big(i(\mathsf{b}_t + \sqrt{\gamma}\omega_t)L\big)\mathsf{I}_{\mathcal{H}}.$$

In particular, U_t commutes with L for all $t \ge 0$. $\diamondsuit$

Towards the Lindblad equation To obtain the Lindblad equation (GKSL) from the stochastic Schrödinger equation (SSE), we consider noise profiles of the form

$$\beta_t^j = \int_0^t u^j(r)\,\mathrm{d}r + \omega_t^j, \qquad j = 1, \dots, d,$$

Now let $\rho_0 \in \mathscr{D}(\mathcal{H})$ be an initial datum for the Lindblad equation and U_t be the solution to (SSE). Then, the $\mathscr{D}(\mathcal{H})$-valued process $\mathsf{a}_t := U_t\,\rho_0\,U_t^\dagger$ satisfies

$$\begin{aligned}
\mathrm{d}\mathsf{a}_t &= \mathrm{d}U_t\,\rho_0\,U_t^\dagger + U_t\,\rho_0\,\mathrm{d}U_t^\dagger + \mathrm{d}U_t\,\rho_0\,\mathrm{d}U_t^\dagger \\
&= (\mathrm{d}\chi_t + (-iH + \mathfrak{I})\,\mathrm{d}t)\,\mathsf{a}_t + \mathsf{a}_t(\mathrm{d}\chi_t^\dagger + (-iH + \mathfrak{I})^\dagger \mathrm{d}t) + \mathrm{d}\chi_t\,\mathsf{a}_t\,\mathrm{d}\chi_t^\dagger \\
&= -i[H_t^u, \mathsf{a}_t]\,\mathrm{d}t + i\sum\nolimits_j [L_j, \mathsf{a}_t]\sqrt{\gamma_j}\,\mathrm{d}\omega_t^j + \tfrac{1}{2}\sum\nolimits_j \gamma_j\big(2L_j\,\mathsf{a}_t\,L_j - L_j^2\mathsf{a}_t - \mathsf{a}_t L_j^2\big)\,\mathrm{d}t \\
&= -i[H_t^u, \mathsf{a}_t]\,\mathrm{d}t + \mathscr{L}(\mathsf{a}_t)\,\mathrm{d}t + i\sum\nolimits_j \sqrt{\gamma_j}[L_j, \mathsf{a}_t]\,\mathrm{d}\omega_t^j,
\end{aligned}$$

where we set the Hamiltonian $H_t^u := H - \sum_j u_j(t)L_j$ and

$$\mathscr{L}(\rho) = \tfrac{1}{2}\sum\nolimits_j \big(2\widehat{L}_j\,\rho\,\widehat{L}_j - \widehat{L}_j^2\rho - \rho\widehat{L}_j^2\big), \qquad \widehat{L}_j := \sqrt{\gamma_j}L_j.$$

Hence, taking the expectation, we obtain for the density operator $\rho_t := \mathbb{E}[\mathsf{a}_t] \in \mathscr{D}(\mathcal{H})$ the Lindblad equation

$$\mathrm{d}\rho_t = -i[H_t^u, \rho_t]\,\mathrm{d}t + \mathscr{L}^*(\rho_t)\,\mathrm{d}t,$$

where we used the fact that $\mathscr{L}^* = \mathscr{L}$ since $L_j \in \mathscr{O}(\mathcal{H})$.

Note, however, that we recover the Lindblad equation with *only* Hermitian jump operators in this way, i.e., $L_j \in \mathscr{O}(\mathcal{H})$. To consider general jump operators, we have to leave the realm of classical noise and talk about quantum noise, which will be the main topic of the remaining sections in this lecture series.

Remark 1.5 We note that if $\rho_0 \in \mathscr{P}(\mathcal{H})$ is a pure state, then a_t is a $\in \mathscr{P}(\mathcal{H})$-valued process, i.e., a_t is almost surely a pure state for all times. ♢

1.4 Example: 1-qubit fidelity of the Hadamard gate

In quantum computing, one is often interested in the *fidelity* of a quantum gate, i.e., a unitary operation, where the fidelity of two density operators $\rho, \sigma \in \mathscr{D}(\mathcal{H})$ is defined by

$$\mathsf{F}(\rho, \sigma) := \mathrm{tr}\Big[(\sqrt{\rho}\sigma\sqrt{\rho})^{\frac{1}{2}}\Big]^2.$$

When, $\rho = |\psi\rangle\langle\psi| \in \mathscr{P}(\mathcal{H})$ is a pure state, then the fidelity reduces to

$$\mathsf{F}(\rho, \sigma) = \langle\psi, \sigma\psi\rangle = \mathrm{tr}[\rho\sigma].$$

If $\sigma = |\varphi\rangle\langle\varphi| \in \mathscr{P}(\mathcal{H})$ is also a pure state, then it simplifies to $\mathsf{F}(\rho, \sigma) = |\langle\psi, \varphi\rangle|^2$, which is much simpler than the case for general density operators. Let's see how the SSE can be used to compute the fidelity of a quantum gate.

Say, we would like to implement a Hadamard gate on a single qubit on the Hilbert space $\mathcal{H} = \mathbb{C}^2 = \mathrm{span}\{\mathbf{e}_0, \mathbf{e}_1\}$. The Hadamard gate is given by

$$U_{\mathsf{h}} = \frac{|\mathbf{e}_0\rangle + |\mathbf{e}_1\rangle}{\sqrt{2}}\langle\mathbf{e}_0| + \frac{|\mathbf{e}_0\rangle - |\mathbf{e}_1\rangle}{\sqrt{2}}\langle\mathbf{e}_1| = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix},$$

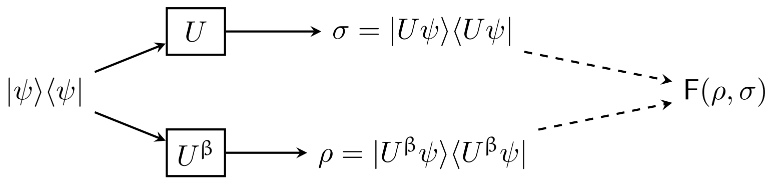

Figure 1: Computing the fidelity of a quantum gate

with the corresponding eigenpairs $(1, \mathbf{e}_+)$ and $(-1, \mathbf{e}_-)$, where

$$\mathbf{e}_+ = \frac{\mathbf{e}_0 + \mathbf{e}_1}{\sqrt{2}}, \qquad \mathbf{e}_- = \frac{\mathbf{e}_0 - \mathbf{e}_1}{\sqrt{2}}.$$

By spectral calculus, we obtain

$$\begin{aligned} U_{\mathsf{h}} &= (1)|\mathbf{e}_+\rangle\langle\mathbf{e}_+| + (-1)|\mathbf{e}_-\rangle\langle\mathbf{e}_-| = e^0|\mathbf{e}_+\rangle\langle\mathbf{e}_+| + e^{-i\pi}|\mathbf{e}_-\rangle\langle\mathbf{e}_-| \\ &= \exp(-i\pi|\mathbf{e}_-\rangle\langle\mathbf{e}_-|) = e^{-i\pi H_{\mathsf{h}}}, \qquad H_{\mathsf{h}} \coloneqq |\mathbf{e}_-\rangle\langle\mathbf{e}_-| = \tfrac{1}{2}(\mathsf{I}_{\mathcal{H}} - \sigma_{\mathsf{x}}). \end{aligned}$$

In other words, $iH_{\mathsf{h}} \in \mathscr{A}(\mathcal{H})$ generates the Hadamard gate after evolving for time $t = \pi$, i.e., $U_t = \exp(-itH_{\mathsf{h}})$ solves the Schrödinger equation

$$\mathrm{d}U_t = -iH_{\mathsf{h}}U_t\,\mathrm{d}t, \qquad U_0 = \mathsf{I}_{\mathcal{H}}.$$

Notice that $H_{\mathsf{h}} \in \mathscr{P}(\mathcal{H})$ happens to be a rank-1 projection on the unit vector $\mathbf{e}_- \in \mathcal{H}$.

In the presence of noise, however, we instead have

$$\mathrm{d}U_t^{\beta} = -iH_{\mathsf{h}}U_t^{\beta} \circ \mathrm{d}\beta_t, \qquad U_0 = \mathsf{I}_{\mathcal{H}},$$

From Remark 1.4, we obtain the explicit form $U_t^{\beta} = \exp(-i\beta_t H_{\mathsf{h}})$.

Suppose we have a noisy pulse $\beta_t = t + \sqrt{\gamma}\omega_t$. Then the fidelity between the desired pure state $|U_t\psi\rangle\langle U_t\psi|$ and the noisy pure state $|U_t^{\beta}\psi\rangle\langle U_t^{\beta}\psi|$ for any unit vector $\psi \in \mathcal{H}$ is

$$\mathsf{F}_t^{\beta}(\psi) \coloneqq |\langle U_t^{\beta}\psi, U_t\psi\rangle|^2 = |\langle U_t^{\dagger}U_t^{\beta}\psi, \psi\rangle|^2 \quad \in [0,1].$$

Using the formulas obtained above, we easily deduce that

$$U_t^{\dagger}U_t^{\beta} = \exp\big(-i\sqrt{\gamma}\omega_t H_{\mathsf{h}}\big) = |\mathbf{e}_+\rangle\langle\mathbf{e}_+| + e^{-i\sqrt{\gamma}\omega_t}|\mathbf{e}_-\rangle\langle\mathbf{e}_-|.$$

Since $\{\mathbf{e}_+, \mathbf{e}_-\}$ forms an orthonormal basis for $\mathcal{H}$, $\psi = \psi_+\mathbf{e}_+ + \psi_-\mathbf{e}_-$, for which we obtain

$$\begin{aligned} \mathsf{F}_t^{\beta}(\psi) &= |\langle \psi_+\mathbf{e}_+ + \psi_- e^{-i\sqrt{\gamma}\omega_t}\mathbf{e}_-, \psi_+\mathbf{e}_+ + \psi_-\mathbf{e}_-\rangle|^2 \\ &= ||\psi_+|^2 + |\psi_-|^2 e^{i\sqrt{\gamma}\omega_t}|^2 = 1 - 2\big(1 - \cos(\sqrt{\gamma}\omega_t)\big)|\psi_+|^2|\psi_-|^2. \end{aligned}$$

From this, we can deduce all statistical properties of the fidelity, e.g., its expectation

$$\mathbb{E}[\mathsf{F}_t^{\beta}(\psi)] = 1 - 2\big(1 - \mathbb{E}[\cos(\sqrt{\gamma}\omega_t)]\big)|\psi_+|^2|\psi_-|^2 = 1 - 2\big(1 - e^{-\gamma t/2}\big)|\psi_+|^2|\psi_-|^2,$$

and variance

$$\begin{aligned} \mathbb{V}[\mathsf{F}_t^{\beta}(\psi)] &= \mathbb{E}\Big[(\mathsf{F}_t^{\beta}(\psi) - \mathbb{E}[\mathsf{F}_t^{\beta}(\psi)])^2\Big] = 4\mathbb{E}\Big[\big(\cos(\sqrt{\gamma}\omega_t) - e^{-\gamma t/2}\big)^2\Big]|\psi_+|^4|\psi_-|^4 \\ &= 4\big(1 - 2\mathbb{E}[\cos(\sqrt{\gamma}\omega_t)]e^{-\gamma t/2} + e^{-\gamma t}\big)^2|\psi_+|^4|\psi_-|^4 = 4\big(1 - e^{-\gamma t}\big)^2|\psi_+|^4|\psi_-|^4, \end{aligned}$$

where we used the fact that $\mathbb{E}[\cos(\sqrt{\gamma}\omega_t)] = \exp(-\gamma t/2)$.

Exercise 1.1 Compute the fidelity of the Hadamard gate using the Lindblad equation.

Question: Can we improve the fidelity by implementing a different pulse?

In [3], we introduced a variational quantum algorithm to do just this, where the fidelity was used as a *regularizer* to a deterministic optimal control problem. For instance, given a desired pure state $|\psi_{\mathsf{d}}\rangle\langle\psi_{\mathsf{d}}| \in \mathscr{P}(\mathcal{H})$ at time $t = T$, one solves the problem

$$\min_{\mathsf{b}} \left\{ |\langle \psi_{\mathsf{d}}, U_T\, \mathbf{e}_0 \rangle|^2 - \lambda \int_0^T \mathbb{E}[\mathsf{F}_t^{\beta}(\mathbf{e}_0)]\, \mathrm{d}t \right\},$$

subject to the deterministic Schrödinger equation

$$\mathrm{d}U_t = -iHU_t\, \mathrm{d}\mathsf{b}_t, \qquad U_0 = \mathsf{I}_{\mathcal{H}},$$

and where

$$\mathbb{E}[\mathsf{F}_t^{\beta}(\mathbf{e}_0)] = \tfrac{1}{2}\big(1 + \mathbb{E}[\cos(\mathsf{b}_t + \sqrt{\gamma}\omega_t)]\big).$$

Here, one is interested in determining a gate U_T that maps $\mathbf{e}_0$ to ψ_{d}, while keeping the fidelity expectation of the evolution under control.

2 Quantum Noise and Fock Space

As shown in the previous section, SSEs driven by classical noise provide a useful class of stochastic unravellings of Lindblad dynamics. However, this approach is fundamentally limited: it produces only Lindblad generators with *Hermitian* jump operators. To describe general irreversible quantum dynamics—including spontaneous emission, particle loss, and counting processes—we must move beyond classical noise and introduce *quantum noise*.

Recall that in Section 1, the stochastic dilation $\mathsf{a}_t = U_t \rho_0 U_t^\dagger$, led, after taking expectations, to a Lindblad equation with Hermitian noise operators $L_j^\dagger = L_j$. While such Lindblad operators model dephasing and diffusion-type noise, they cannot describe dissipative processes such as amplitude damping, where the jump operator is non-Hermitian. An important example is the jump operator $L = \sigma_- = |0\rangle\langle 1|$ on a single qubit describing *spontaneous emission* due to black body radiation.

From a physical perspective, this reflects the fact that classical noise models only randomize *phases* or *energies*. Truly quantum processes involve the exchange of quanta with an environment, and therefore require a non-commutative noise model.

Unitary dilations and infinite environments

A guiding principle in the theory of open quantum systems is that irreversibility arises from neglecting environmental degrees of freedom in the following sense. One considers the composite Hilbert space $\mathcal{H}_{\text{tot}} = \mathcal{H}_{\text{sys}} \otimes \mathcal{H}_{\text{env}}$, describing the *universe* and consists of the system $\mathcal{H}_{\text{sys}}$ and an environment $\mathcal{H}_{\text{env}}$. In this setting, one obtains a unitary evolution

$$U_t : \mathcal{H}_{\text{tot}} \to \mathcal{H}_{\text{tot}}$$

The reduced evolution of the system state is then obtained by tracing out the environment,

$$\rho_t = \operatorname{tr}_{\mathcal{H}_{\text{env}}} \left[U_t (\rho_0 \otimes \rho_{\text{env}}) U_t^\dagger \right] \in \mathscr{D}(\mathcal{H}_{\text{sys}}),$$

where $\rho_0 \in \mathscr{D}(\mathcal{H}_{\text{sys}})$ is the initial state on the system and $\rho_{\text{env}} \in \mathscr{D}(\mathcal{H}_{\text{env}})$ is a given state on the environment. Requiring *Markovianity* and time-homogeneity of the evolution inevitably forces the environment to possess infinitely many degrees of freedom. In continuous time, this naturally leads to a description in terms of bosonic quantum fields.

2.1 Bosonic Fock space

The *bosonic (or symmetric) Fock space* of a complex Hilbert space $\mathcal{K}$ is defined as

$$\mathfrak{F}(\mathcal{K}) := \bigoplus_{n \in \mathbb{N}_0} \mathcal{K}^{\odot n},$$

where $\mathcal{K}^{\odot n}$ denotes the n-fold symmetric tensor product, i.e.,

$$f \in \mathcal{K}^{\odot n} \quad \Leftrightarrow \quad f(\sigma(u_1, \ldots, u_n)) = f(u_1, \ldots, u_n) \quad \text{for any permutation } \sigma,$$

and by convention $\mathcal{K}^{\odot 0} := \mathbb{C}$. The bosonic Fock space $\mathfrak{F}(\mathcal{K})$ inherits the scalar product from $\mathcal{K}$ defined by

$$\langle \oplus u^{(n)}, \oplus v^{(n)} \rangle_{\mathfrak{F}(\mathcal{K})} := \sum_{n \in \mathbb{N}_0} \langle u^{(n)}, v^{(n)} \rangle_{\mathcal{K}^{\otimes n}}.$$

We define the *exponential vectors*

$$\mathsf{e}(u) = \bigoplus_{n\in\mathbb{N}_0} \frac{1}{\sqrt{n!}} u^{\otimes n}, \qquad u \in \mathcal{K},$$

and the distinguished vector $\Omega := \mathsf{e}(0) = 1 \oplus 0 \oplus \cdots \in \mathfrak{F}(\mathcal{K})$, called the *vacuum vector.*

It turns out that the set of exponential vectors $\mathfrak{E}(\mathcal{K})$ is *total* in $\mathfrak{F}(\mathcal{K})$, i.e., the linear span of $\mathfrak{E}(\mathcal{K})$ is dense in $\mathfrak{F}(\mathcal{K})$. This fact will be helpful for us in the future. Since,

$$\langle \mathsf{e}(u), \mathsf{e}(v)\rangle_{\mathfrak{F}(\mathcal{K})} = \sum_{n\in\mathbb{N}_0} \frac{1}{n!} \langle u, v\rangle_{\mathcal{K}}^n = e^{\langle u,v\rangle_{\mathcal{K}}}, \qquad u, v \in \mathcal{K},$$

the exponential vectors are normalizable. These normalized exponential vectors

$$\psi(u) = e^{-\frac{1}{2}\|u\|_{\mathcal{K}}^2} \mathsf{e}(u), \qquad u \in \mathcal{K},$$

are called *coherent vectors.*

For time-continuous noise, the *canonical* choice is

$$\mathcal{K} = L^2(\mathbb{R}_+, \mathsf{N}; \mathbb{C}^d) \cong L^2(\mathbb{R}_+, \mathsf{N}) \otimes \mathbb{C}^d,$$

where $d \in \mathbb{N}$ represents the number of *noise channels* and N is the Lebesgue measure. From now on, we will only consider this choice with $d = 1$ and call $\mathfrak{F}(\mathcal{K})$ our noise environment.

2.2 Creation, annihilation, and gauge processes

On the bosonic Fock space $\mathfrak{F}(\mathcal{K})$, one defines operator-valued processes on $\mathfrak{E}(\mathcal{K})$:

the *annihilation* processes $\mathsf{A}(t)$, the *creation* processes $\mathsf{A}^\dagger(t)$,
the *gauge (or counting)* processes $\mathsf{N}(t)$.

Heuristically, $\mathsf{A}(t)$ annihilates a quantum in channel j arriving before time $t \geq 0$, $\mathsf{A}^\dagger(t)$ creates such a quantum, and $\mathsf{N}(t)$ counts quanta in the channels. Together, these processes encode absorption, emission, and counting statistics in a unified operator-theoretic framework and form the building blocks of quantum stochastic calculus, serving as driving noises in the Hudson-Parthasarathy theory of quantum processes.

Definition 2.1 Let $\mathcal{D}$ be a total subset of a complex Hilbert space $\mathcal{H}$.

(i) A *random variable* χ is an element of $\mathscr{L}(\mathcal{D}; \mathcal{H})$, where

$$\mathscr{L}(\mathcal{D}; \mathcal{H}) := \Big\{ \mathsf{Z} : \mathcal{D} \to \mathcal{H} \ \text{ linear} : \mathcal{D} \subset \operatorname{dom}(\mathsf{Z}) \cap \operatorname{dom}(\mathsf{Z}^\dagger) \Big\}.$$

(ii) A *stochastic process* in $\mathcal{H}$ is a family $(\chi(t))_{t\in\mathbb{R}_+}$ of random variables such that

$$\mathbb{R}_+ \ni t \mapsto \chi(t)\eta \quad \text{is Borel measurable for every } \eta \in \mathcal{D}.$$

(iii) A *martingale* is a map $\mathbb{R}_+ \ni t \mapsto \mathtt{m}_t \in \mathcal{K}$ satisfying

$$\mathtt{m}_t \in L^2([0,t], \mathsf{N}) \qquad \text{and} \qquad \mathbf{1}_{[0,s]}\mathtt{m}_t = \mathtt{m}_s \quad \text{for every } s \leq t.$$

Example 2.2 A simple martingale is $\mathtt{m}_t = \mathbf{1}_{[0,t]}$, which we shall call the *canonical* martingale. Other martingales include $\mathtt{m}_t = v\mathbf{1}_{[0,t]}$, where $v \in \mathcal{K}$ is an arbitrary function.

Notice that nothing about the definition above is stochastic in the usual sense.

Annihilation and creation processes. The annihilation and creation operators corresponding to a martingale $\mathtt{m}$ are defined on exponential vectors $\mathsf{e}(u)$, $u \in \mathcal{K}$ by

$$\mathsf{A}_{\mathtt{m}}(t)\mathsf{e}(u) := \langle \mathtt{m}_t, u\rangle \mathsf{e}(u), \qquad \mathsf{A}^{\dagger}_{\mathtt{m}}(t)\mathsf{e}(u) := \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\Big|_{\varepsilon=0} \mathsf{e}(u + \varepsilon \mathtt{m}_t).$$

These operators are densely defined, mutually adjoint, and they satisfy the so-called *canonical commutation relations*

$$[\mathsf{A}_{\mathtt{m}}(t), \mathsf{A}^{\dagger}_{\mathtt{m}}(s)] = \langle \mathtt{m}_t, \mathtt{m}_s\rangle\, \mathsf{I}_{\mathfrak{F}(\mathcal{K})}, \qquad [\mathsf{A}_{\mathtt{m}}(t), \mathsf{A}_{\mathtt{m}}(s)] = [\mathsf{A}^{\dagger}_{\mathtt{m}}(t), \mathsf{A}^{\dagger}_{\mathtt{m}}(s)] = 0. \tag{CCR}$$

Due to the martingale property of $\mathtt{m}$, we also have that

$$(\mathsf{A}_{\mathtt{m}}(t) - \mathsf{A}_{\mathtt{m}}(s))\mathsf{e}(u) = \langle \mathtt{m}_t - \mathtt{m}_s, u\rangle \mathsf{e}(u) = \langle \mathbf{1}_{[s,t]}\mathtt{m}_t, u\rangle \mathsf{e}(u).$$

These relations give rise to the quantum Itô table that we will see in the following section.

With respect to the coherent vector $\psi(u) \in \mathfrak{F}(\mathcal{K})$, $u \in \mathcal{K}$, one has

$$\langle \psi(u), \mathsf{A}_{\mathtt{m}}(t)\psi(u)\rangle = \langle \mathtt{m}_t, u\rangle = \overline{\langle \psi(u), \mathsf{A}^{\dagger}_{\mathtt{m}}(t)\psi(u)\rangle},$$
$$\langle \psi(u), [\mathsf{A}_{\mathtt{m}}(t), \mathsf{A}^{\dagger}_{\mathtt{m}}(s)]\psi(u)\rangle = \langle \mathtt{m}_t, \mathtt{m}_s\rangle.$$

In particular, in the vacuum vector $\Omega = \psi(0)$, we find

$$\langle \Omega, \mathsf{A}_{\mathtt{m}}(t)\Omega\rangle = \overline{\langle \Omega, \mathsf{A}^{\dagger}_{\mathtt{m}}(t)\Omega\rangle} = 0, \qquad \langle \Omega, \mathsf{A}_{\mathtt{m}}(t)\mathsf{A}^{\dagger}_{\mathtt{m}}(s)\Omega\rangle = \langle \mathtt{m}_t, \mathtt{m}_s\rangle,$$

which shows that the self-adjoint field operators $\mathsf{B}_{\mathtt{m}} = \mathsf{A}_{\mathtt{m}} + \mathsf{A}^{\dagger}_{\mathtt{m}}$ reproduce the covariance structure of classical Brownian motion for the canonical martingale $\mathtt{m}_t = \mathbf{1}_{[0,t]}$, i.e.,

$$\langle \Omega, \mathsf{B}_{\mathtt{m}}(t)\mathsf{B}^{\dagger}_{\mathtt{m}}(s)\Omega\rangle = t \wedge s.$$

Thus, classical noise is recovered as a commutative subtheory of quantum noise.

Exercise 2.1 Use the *Zassenhaus formula* and the fact that

$$[\mathsf{A}_{\mathtt{m}}(t), [\mathsf{A}_{\mathtt{m}}(t), \mathsf{A}^{\dagger}_{\mathtt{m}}(t)]] = [\mathsf{A}^{\dagger}_{\mathtt{m}}(t), [\mathsf{A}_{\mathtt{m}}(t), \mathsf{A}^{\dagger}_{\mathtt{m}}(t)]] = 0 \qquad \text{for all } t \geq 0,$$

to show that for every $r \in \mathbb{R}$,

$$\mathbb{E}_{\Omega}\big[e^{ir\mathsf{B}_{\mathtt{m}}(t)}\big] := \langle \Omega, e^{ir\mathsf{B}_{\mathtt{m}}(t)}\Omega\rangle = \exp\Big(-\tfrac{1}{2}r^2\|\mathtt{m}_t\|^2_{\mathcal{K}}\Big).$$

Conclude from this that, under the vacuum vector Ω, the field operator $\mathsf{B}(t)$ is a Gaussian *random variable* with mean 0 and variance $\Sigma = \|\mathtt{m}_t\|^2$.

What would change if we replace the vacuum Ω with a coherent vector $\psi(u)$?

Gauge processes. The gauge processes $\mathsf{N}(t)$ *counts* the number of quanta in the channel up to time $t \geq 0$. Their expectation on coherent vectors $\psi(u)$ is given by

$$\langle \psi(u), \mathsf{N}(t)\psi(u)\rangle = \int_{[0,t]} \overline{u}(s)u(s)\,\mathrm{d}s = \|u\|^2_{\mathcal{K}}.$$

In particular, its distribution is given by

$$\mathbb{E}_{\psi(u)}\big[e^{ir\mathsf{N}(t)}\big] := \langle \psi(u), e^{ir\mathsf{N}(t)}\psi(u)\rangle = \exp\Big((e^{ir} - 1)\|u\|^2_{\mathcal{K}}\Big),$$

i.e., in the coherent state $\psi(u)$, $\mathsf{N}(t)$ has a Poisson statistics with intensity $|u|^2\mathsf{N}$.

In these lectures, we only have time to focus on the creation and annihilation processes. A proper treatise of the gauge process requires more preparation, but its inclusion in the following theory may be done without too much trouble.

2.3 Fock-Wiener isometry

The Wiener-Itô-Segal isomorphism provides a precise mathematical link between classical and quantum noise, which identifies the canonical bosonic Fock space with an L^2-space over classical Wiener space.

More precisely, let $(\mathcal{C}(\mathbb{R}_+;\mathbb{R}), \mathcal{F}, \mathbb{W})$ be a classical Wiener probability space with canonical process $(X_t)_{t\in\mathbb{R}_+}$ with $X_t(\omega) = \omega(t)$, $\omega \in \mathcal{C}(\mathbb{R}_+;\mathbb{R})$. Then there exists a unitary isomorphism

$$\mathcal{U} : \mathfrak{F}(\mathcal{K}) \to L^2(\mathcal{C}(\mathbb{R}_+;\mathbb{R}), \mathbb{W}),$$

called the *Fock-Wiener (or Wiener-Itô-Segal) isometry*, with the following properties:

(i) The vacuum vector Ω is mapped to the constant function 1.

(ii) Exponential vectors correspond to stochastic exponentials of Brownian motion, i.e.,

$$\mathcal{U}\big(\psi(u\mathbf{1}_{[0,t]})\big)(\omega) = \exp\bigg(\int_0^t u(s)\,\mathrm{d}X_s(\omega) - \frac{1}{2}\|u\mathbf{1}_{[0,t]}\|_{\mathcal{K}}^2\bigg) \quad \text{for } t \geq 0.$$

Recall that the right-hand side is an exponential martingale and that the family of such functions is total in $L^2(\mathcal{C}(\mathbb{R}_+;\mathbb{R}), \mathbb{W})$.

(iii) Multiple Wiener integrals of order n correspond to the n-particle sector $\mathcal{K}^{\odot n}$.

Under this isometry, the self-adjoint field operator $\mathsf{B}(t)$ acts as multiplication by the classical Brownian motion W_t. In particular,

$$\mathcal{U}\mathsf{B}(t)\mathcal{U}^{-1} = W_t,$$

viewed as a multiplication operator on the commutative algebra $L^\infty(\mathcal{C}(\mathbb{R}_+;\mathbb{R}), \mathbb{W})$. This identification shows that classical stochastic calculus is faithfully embedded into quantum stochastic calculus as the restriction to a commuting subalgebra of field operators.

Remark 2.3 (1) Notice that the coherent states $\psi(u) \in \mathfrak{F}(\mathcal{K})$ play the role of changing the reference measure $\mathbb{W}$ by the drift field $u \in \mathcal{K}$.

(2) A similar construction holds for Poisson processes on the Skorokhod space. ♢