Additional information and pricing-hedging duality in robust framework

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based on joint work with

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Pricing and hedging problems

How to determine the prices of exotic options?

How to hedge the positions in exotic options by using underlying assets and vanilla options?

- Model-specific approach: the price process of the underlying assets $(S_t)_{t \leq T}$ are modelled by some parametric family of stochastic processes.
- Model-independent approach: many possible models, weaker economic assumptions
  - Quasi-sure approach
  - Pathwise approach

Robust approach – an active field of research

Explicit bounds $LB \leq \mathcal{P}O_T \leq UB$ *and robust super-/sub- hedges*

Arbitrage considerations and robust FTAP

Pricing-hedging duality

Pathspace restrictions $A \subsetneq \Omega$

Acciaio, Bayraktar, Beiglböck, Biagini, Bouchard, Brown, Burzoni, Cheridito, Cox, Davis, Denis, Dolinsky, Dupire, Frittelli, Galichon, Gassiat, Guo, Henry-Labordère, Hobson, Hou, Huesmann, Källblad, Kardaras, Klimmek, Kupper, Maggis, Martini, Mykland, Nadtochiy, Neuberger, Neufeld, Nutz, Obłój, Penker, Perkowski, Possamaï, Prömel, Raval, Riedel, Rogers, Schachermayer, Soner, Spoida, Tan, Tangpi, Temme, Touzi, Wang ...

Additional information

- $\mathbb{F}$ regular agent/ common knowledge/ public information
- $\mathbb{G} \supset \mathbb{F}$ informed agent with additional information represented by the entire filtration
- Hedging ξ: how much the additional information is worth?

$$\text{Price}_{\mathbb{F}}(\xi) - \text{Price}_{\mathbb{G}}(\xi)$$

- Super-hedging price:

$$\inf\{x : \exists (x, \gamma)\text{-super-hedges } \xi\}$$

vs market model price:

$$\sup_{\mathbb{P} \in \mathcal{M}} \mathbb{E}_{\mathbb{P}}(\xi)$$

- No (duality) gap between these two values

Set-up

- Yan Dolinsky and Mete Soner *Martingale Optimal Transport and Robust Hedging in Continuous Time*, Probability Theory and Related Fields. 160. (2014), 391–427.
- Zhaoxu Hou and Jan Obłój *On robust pricing-hedging duality in continuous time*.

Traded assets and information

- Stock price S is the canonical process on

$$\Omega := \{\omega \in C([0,T], \mathbb{R}_+^d) : \omega(0) = 1\}$$

- $\mathbb{F}$ is the filtration generated by S, i.e., $\mathcal{F}_t := \sigma(S_s : s \leq t)$
- $X_0, X_1, ..., X_n$ statically traded options which have prices $\mathcal{P}(X_i)$ at time 0, $X_0 = 1$ and $\mathcal{P}(X_0) = 1$
- $\mathbb{G}$ is the enlarged filtration $\mathcal{G}_t := \mathcal{F}_t \vee \mathcal{H}_t$, where $\mathbb{H} := (\mathcal{H}_t)_{t \leq T}$ is another filtration
  $\mathbb{G}$ is called the initial enlargement of $\mathbb{F}$ with random variable Z if $\mathcal{H}_t = \sigma(Z)$

Equivalence relation and atoms

- $(\Omega, \mathcal{F}_T)$ measurable space and $\mathcal{G}$ sub-σ-field of $\mathcal{F}_T$, $\omega, \widetilde{\omega} \in \Omega$

  ω and $\widetilde{\omega}$ are $\mathcal{G}$-equivalent, $\omega \sim_{\mathcal{G}} \widetilde{\omega}$, if $\mathbb{1}_G(\omega) = \mathbb{1}_G(\widetilde{\omega})$ holds $\forall\ G \in \mathcal{G}$

  Note that $\omega \sim_{\mathcal{F}_t} \widetilde{\omega} \iff \omega_u = \widetilde{\omega}_u$ for each $u \leq t$,
  and $\omega \sim_{\sigma(Z)} \widetilde{\omega} \iff Z(\omega) = Z(\widetilde{\omega})$
- $[\omega]_{\mathcal{G}}$ denotes the equivalence class, or atom, in Ω where ω belongs to:

$$[\omega]_{\mathcal{G}} = \bigcap \{A : A \in \mathcal{G}, \omega \in A\}$$

- $(\Omega, \mathcal{G})$ is countably generated if there exists a *sequence* $(B_n)_{n \in \mathbb{N}} \subset \mathcal{G}$ such that $\sigma((B_n)_{n \in \mathbb{N}}) = \mathcal{G}$. In this case each atom is $\mathcal{G}$-measurable

Trading strategies

- integral of $g : [0, T] \to \mathbb{R}^d$ of finite variation w.r.t. $\omega \in \Omega$:

$$\int_0^t g(u)d\omega(u) := g(t)\omega(t) - g(0)\omega(0) - \int_0^t \omega(u)dg(u)$$

- $\gamma : \Omega \to \mathcal{D}([0, T], \mathbb{R}^d)$ is $\mathbb{G}$-adapted if γ_t is $\mathcal{G}_t$-measurable, i.e. if

$$\omega \sim_{\mathcal{G}_t} \widetilde{\omega} \quad \text{implies} \quad \gamma(\omega)_t = \gamma(\widetilde{\omega})_t$$

  and it is $\mathbb{G}$-admissible strategy if moreover it has finite variation and

$$\int_0^t \gamma(\omega)_u dS_u(\omega) \geq -M(\omega) \quad \forall \omega, t \quad \text{for some } M \in L^0(\Omega, \mathcal{G}_0)$$

- A $\mathbb{G}$-admissible semi-static strategy is a pair (X, γ) where $X = A_0 + \sum_{i=1}^n A_i X_i$ for some $\mathcal{G}_0$-measurable random variables A_i and $\mathbb{G}$-admissible strategy γ.
  Initial cost of such a strategy is $\mathcal{P}(X) = A_0 + \sum_{i=1}^m A_i \mathcal{P}(X_i)$.
  The set of all $\mathbb{G}$-admissible semi-static strategies is denoted by $\mathcal{A}(\mathbb{G})$.

The super-hedging price

$\mathbb{G}$-super-hedging price of ξ on $A \in \mathcal{F}_T$:

$$V_A^{\mathbb{G}}(\xi)(\omega) := \inf\{\mathcal{P}(X)(\omega) : \quad \exists\,(X,\gamma) \in \mathcal{A}(\mathbb{G}) \text{ such that } X(\widetilde{\omega}) + \int_0^T \gamma(\widetilde{\omega})_u dS_u(\widetilde{\omega}) \geq \xi(\widetilde{\omega}) \text{ for all } \widetilde{\omega} \in A\}$$

Proposition

The $\mathbb{G}$-super-hedging price on Ω is constant on each $[\omega]$ and given by

$$V_\Omega^{\mathbb{G}}(\xi)(\omega) = V_{[\omega]}^{\mathbb{G}}(\xi)$$

where $[\omega]$ denotes the $\mathcal{G}_0$-equivalence class containing ω.

It holds that $\quad V_\Omega^{\mathbb{G}}(\xi) \leq V_\Omega^{\mathbb{F}}(\xi)$

The market model price

- The set of $\mathbb{G}$-calibrated martingale measures concentrated on $A \in \mathcal{F}_T$:

$$\mathcal{M}_A^{\mathbb{G}} := \{\mathbb{P} : S \text{ is a } (\mathbb{P}, \mathbb{G})\text{-martingale}, \mathbb{P}(A) = 1 \\ \text{and } \mathbb{E}_{\mathbb{P}}(X_i | \mathcal{G}_0) = \mathcal{P}(X_i) \text{ for all } i \in \{1, ..., n\} \ \mathbb{P}\text{-a.s.}\}$$

- $\mathbb{G}$-market price of ξ on $A \in \mathcal{F}_T$:

$$P_A^{\mathbb{G}}(\xi)(\omega) := \text{"}\sup_{\mathbb{P} \in \mathcal{M}_A^{\mathbb{G}}} \mathbb{E}_{\mathbb{P}}(\xi | \mathcal{G}_0)(\omega)\text{"}$$

Proposition

Assume each element of $\mathbb{G}$ is countably generated. Let $\mathbb{P} \in \mathcal{M}_{\mathcal{X},\mathcal{P},\Omega}^{\mathbb{G}}$. Then, there exists a set $\Omega^{\mathbb{P}} \in \mathcal{G}_0$ with $\mathbb{P}(\Omega^{\mathbb{P}}) = 1$ and a version $\{\mathbb{P}_\omega\}$ of the regular conditional probabilities of $\mathbb{P}$ with respect to $\mathcal{G}_0$ such that for each $\omega \in \Omega^{\mathbb{P}}$, $\mathbb{P}_\omega \in \mathcal{M}_{\mathcal{X},\mathcal{P},[\omega]_{\mathcal{G}_0}}^{\mathbb{G}}$.

The market model price

Let $A \in \mathcal{F}_T$. The $\mathbb{G}$-market price of ξ on A is defined by

$$P_A^{\mathbb{G}}(\xi)(\omega) := \sup_{\mathbb{P} \in \mathcal{M}^{\mathbb{G}}_{\mathcal{X},\mathcal{P},A}} \bar{\mathbb{E}}_{\mathbb{P}_\omega}(\xi), \quad \omega \in \Omega,$$

where $\bar{\mathbb{E}}_{\mathbb{P}_\omega}(\xi) = \mathbb{E}_{\mathbb{P}_\omega}(\xi)$ for $\omega \in \Omega^{\mathbb{P}}$ and $\bar{\mathbb{E}}_{\mathbb{P}_\omega}(\xi) = -\infty$ for $\omega \in \Omega \backslash \Omega^{\mathbb{P}}$.

Proposition

The $\mathbb{G}$-market price on Ω is constant on each $[\omega]$ and given by

$$P_\Omega^{\mathbb{G}}(\xi)(\omega) = P_{[\omega]}^{\mathbb{G}}(\xi)$$

where $[\omega]$ is $\mathcal{G}_0$-equivalent class containing ω.

It holds that $P_\Omega^{\mathbb{G}}(\xi) \leq P_\Omega^{\mathbb{F}}(\xi)$

Easy inequality in the pricing-hedging duality

Lemma

The $\mathbb{G}$-super-hedging price $V_\Omega^{\mathbb{G}}(\xi)$ and the $\mathbb{G}$-market model price $P_\Omega^{\mathbb{G}}(\xi)$ of ξ on Ω satisfy

$$V_\Omega^{\mathbb{G}}(\xi)(\omega) \geq P_\Omega^{\mathbb{G}}(\xi)(\omega) \quad \forall \omega \in \Omega \,.$$

PROOF: $\mathbb{G}$-super-replicating portfolio $(X, \gamma) \in \mathcal{A}^M(\mathbb{G})$ on $[\omega]_{\mathcal{G}_0}$ and measure $\mathbb{P} \in M^{\mathbb{G}}_{[\omega]_{\mathcal{G}_0}}$.
$\{\mathbb{P}_v\}$ regular conditional probabilities of $\mathbb{P}$ with respect to $\mathcal{G}_0$
Since $\mathbb{P}(M \equiv const) = 1$,

$$\mathbb{E}_{\mathbb{P}}(\xi) \leq \mathbb{E}_{\mathbb{P}}\left(X + \int_0^T \gamma_u dS_u\right) \leq \mathbb{E}_{\mathbb{P}}(X).$$

□

Duality

Atoms and path restriction

- Let $\mathbb{G} = \mathbb{F} \vee \sigma(Z)$
- Additional information arrives entirely at time 0
- Atoms of $\mathcal{G}_0$ are simply atoms of $\sigma(Z)$
- Each atom can be seen as path restriction since on each atom the filtration $\mathbb{G}$ and $\mathbb{F}$ coincide, i.e., for each ω

$$\forall G \in \mathcal{G}_t \quad \exists F \in \mathcal{F}_t \quad \text{s.t.} \quad [\omega]_{\mathcal{G}_0} \cap G = [\omega]_{\mathcal{G}_0} \cap F$$

Duality for $\mathbb{G} = \mathbb{F} \vee \sigma(Z)$

Theorem

Let $\mathbb{G} = \mathbb{F} \vee \sigma(Z)$ and assume that for each value c of Z we have

$$P^{\mathbb{F}}_{\{Z=c\}}(\xi) = V^{\mathbb{F}}_{\{Z=c\}}(\xi)$$

for any bounded uniformly continuous ξ.
Suppose that assumptions of Theorem Hou Obłój are satisfied.
Then, duality in $\mathbb{G}$ holds, i.e.,

$$V^{\mathbb{G}}_{\Omega}(\xi)(\omega) = P^{\mathbb{G}}_{\Omega}(\xi)(\omega)$$

for any bounded uniformly continuous ξ.

PROOF: One can show that:

$$P^{\mathbb{F}}_{\{Z=c\}}(\xi) = P^{\mathbb{G}}_{\{Z=c\}}(\xi) \leq V^{\mathbb{G}}_{\{Z=c\}}(\xi) = V^{\mathbb{F}}_{\{Z=c\}}(\xi)$$

[HO] Beliefs: approximate pricing-hedging duality

Approximation of A: $A^\varepsilon = \{\omega \in \Omega : \inf_{v \in A} ||\omega - v|| \le \varepsilon\}$.

$$\widetilde{V}_A^{\mathbb{F}}(\xi) := \inf\{\mathcal{P}(X) : \ \exists (X,\gamma) \in \mathcal{A}(\mathbb{F}),\ \varepsilon > 0 \text{ s.t. } X + \int_0^T \gamma_u dS_u \ge \xi \ \text{ on } A^\varepsilon\}$$

$$\widetilde{P}_A(\xi) := \lim_{\varepsilon \searrow 0} \sup_{\mathbb{P} \in \mathcal{M}_A^{\mathbb{F},\varepsilon}} \mathbb{E}_{\mathbb{P}}(\xi) \quad \text{where}$$

$$\mathcal{M}_A^{\mathbb{F},\varepsilon} := \{\mathbb{P} : S \text{ is a } (\mathbb{P},\mathbb{F})\text{-mart.}, \mathbb{P}(A^\varepsilon) = 1 - \varepsilon \text{ and } |\mathbb{E}_{\mathbb{P}}(X_i) - \mathcal{P}(X_i)| \le \varepsilon \forall i\}$$

Theorem (Hou & Obłój)

Assume that all payoffs X_i are bounded and uniformly continuous and that for all $\varepsilon > 0$ there exists $\mathbb{P} \in \mathcal{M}_A^{\mathbb{F},\varepsilon}$.
Then for any bounded uniformly continuous $\xi : \Omega \to \mathbb{R}$

$$\widetilde{V}_A(\xi) = \widetilde{P}_A(\xi).$$

This theorem implies that

$$P_A(\xi) \le V_A(\xi) \le \widetilde{V}_A(\xi) = \widetilde{P}_A(\xi)$$

Duality for $\mathbb{G} = \mathbb{F} \vee \sigma(Z)$

Theorem

Let $\mathbb{G} = \mathbb{F} \vee \sigma(Z)$ and assume that for each value c of Z we have

$$P^{\mathbb{F}}_{\{Z=c\}}(\xi) = \widetilde{P}^{\mathbb{F}}_{\{Z=c\}}(\xi)$$

for any bounded uniformly continuous ξ.
Suppose that assumptions of Theorem Hou Obłój are satisfied.
Then, duality in $\mathbb{G}$ holds, i.e.,

$$V^{\mathbb{G}}_{\Omega}(\xi)(\omega) = P^{\mathbb{G}}_{\Omega}(\xi)(\omega)$$

for any bounded uniformly continuous ξ.

EXAMPLE: Assume no options and $d = 1$. No duality gap in $\mathbb{G}$ holds:

- $Z = \sup_{t\in[0,T]} |\ln S_t|$
- $Z = \mathbb{1}_{\{a<S_t<b\ \forall t\in[0,T]\}}$ where $a < 1 < b$

Dynamic programming principle

Dynamic approach

Additional information $\sigma(Z)$ is disclosed at time $T_1 \in (0, T)$:

$$\mathcal{G}_t = \mathcal{F}_t \text{ for } t \in [0, T_1) \text{ and } \mathcal{G}_t = \mathcal{F}_t \vee \sigma(Z) \text{ for } t \in [T_1, T]$$

Assume Z satisfies

$$Z(\omega) = \begin{cases} \widetilde{Z}\left(\frac{\omega|_{[T_1,T]}}{\omega_{T_1}}\right) & \text{if} \quad \omega_{T_1} > 0 \\ 1 & \text{if} \quad \omega_{T_1} = 0 \end{cases} \quad \text{for r.v. } \widetilde{Z} \text{ on } \Omega|_{[T_1,T]}.$$

This encodes the idea that the additional information concerns only the evolution of prices after time T_1 irrespectively of the prices before time T_1.

Theorem

Duality in $\mathbb{G}$ holds, i.e., $V_\Omega^{\mathbb{G}}(\xi) = P_\Omega^{\mathbb{G}}(\xi)$ holds for any bounded uniformly continuous ξ.

Dynamic approach

- Firstly solve the problem for each atom of $\mathcal{G}_{T_1}$ on $[T_1, T]$ separately by the same arguments as for $\mathbb{F} \vee \sigma(Z)$
- Secondly aggregate atoms of $\mathcal{G}_{T_1}$ into atoms of $\mathcal{F}_{T_1}$
- Apply dynamic principle
- Thus the problem is now reduced to $[0, T_1]$ and trading w.r.t $\mathbb{F}$

$$V_\Omega^{\mathbb{G},[0,T]}(\xi) = V_\Omega^{\mathbb{F},[0,T_1]}(V_\Omega^{\mathbb{G},[T_1,T]}(\xi)) = V_\Omega^{\mathbb{F},[0,T_1]}(P_\Omega^{\mathbb{G},[T_1,T]}(\xi))$$
$$= P_\Omega^{\mathbb{F},[0,T_1]}(P_\Omega^{\mathbb{G},[T_1,T]}(\xi)) = P_\Omega^{\mathbb{G},[0,T]}(\xi).$$

Dynamic programming principle

Proposition

Let ξ be uniformly continuous.
(i) Define

$$V_\Omega^{\mathbb{G},[T_1,T]}(\xi)(\omega) := \inf\{x : \exists\, \gamma \in \mathcal{A}(\mathbb{G})\, s.t.\, x + \int_{T_1}^{T} \gamma_u dS_u \geq \xi \ on\ [\omega]_{\mathcal{F}_{T_1}}\}.$$

Then, $V_\Omega^{\mathbb{G},[T_1,T]}(\xi)$ is u.c. and

$$V_\Omega^{\mathbb{G},[0,T]}(\xi) = V_\Omega^{\mathbb{F},[0,T_1]}\left(V_\Omega^{\mathbb{G},[T_1,T]}(\xi)\right)$$

(ii) Define $P_\Omega^{\mathbb{G},[T_1,T]}(\xi)(\omega) := \sup_{\mathbb{P}\in\mathcal{M}^{\mathbb{G},[T_1,T]}_{[\omega]_{\mathcal{F}_{T_1}}}} \mathbb{E}_{\mathbb{P}}(\xi)$. Then, $P_\Omega^{\mathbb{G},[T_1,T]}(\xi)$ is u.c. and

$$P_\Omega^{\mathbb{G},[0,T]}(\xi) = P_\Omega^{\mathbb{F},[0,T_1]}\left(P_\Omega^{\mathbb{G},[T_1,T]}(\xi)\right)$$

Conclusions

- Formulation of the duality problem for a general filtration with possibly non trivial initial σ-field
- Translating original problem to the path restriction language from Hou & Obłój in case of an initial enlargement
- Disclosure of an additional information after initial time and dynamic programming principle

Thank You!

Dynamic programming principle

The path modification mapping $\alpha^{v,\widetilde{v}}$ by

$$\alpha(\omega) := \begin{cases} v|_{[0,T_1]} \otimes \frac{v_{T_1}}{\widetilde{v}_{T_1}}\omega|_{[T_1,T]} & \omega \in B^{\widetilde{v}} \\ \widetilde{v}|_{[0,T_1]} \otimes \frac{\widetilde{v}_{T_1}}{v_{T_1}}\omega|_{[T_1,T]} & \omega \in B^{v} \\ \omega & \omega \notin B^{v} \cup B^{\widetilde{v}} \end{cases}$$

If the strategy γ super-replicates on B^v, the strategy $\frac{v_{T_1}}{\widetilde{v}_{T_1}}\gamma \circ \alpha + \frac{\delta^{\frac{1}{4}}}{\widetilde{v}_{T_1}}\mathbb{1}_{[T_1,\widetilde{\tau})}$ super-replicates on $B^{\widetilde{v}}$ and $\widetilde{\xi}(\widetilde{v}) \leq \widetilde{\xi}(v) + e_{\widetilde{\xi}}(||v - \widetilde{v}||)$.

If $\mathbb{P} \in \mathcal{M}_{B^v}^{\mathbb{G},[T_1,T]}$ then $\bar{\mathbb{P}} = \mathbb{P} \circ \alpha \in \mathcal{M}_{B^v}^{\mathbb{G},[T_1,T]}$ and $|\mathbb{E}_{\mathbb{P}}(\xi) - \mathbb{E}_{\bar{\mathbb{P}}}(\xi)| \leq e_{\widehat{\xi}}(||v - \widetilde{v}||)$

Informational metrics

Quantification of the value of the information

Initial enlargement – distances between σ-fileds $d(\mathcal{G},\mathcal{H})$:

$$\sup_{G\in\mathcal{G}} \inf_{H\in\mathcal{H}} \sup_{\mathbb{P}\in\mathcal{M}} \mathbb{P}(G\Delta H) \vee \sup_{H\in\mathcal{H}} \inf_{G\in\mathcal{G}} \sup_{\mathbb{P}\in\mathcal{M}} \mathbb{P}(G\Delta H)$$

$$\sup_{0\leq\xi\leq 1} \sup_{\mathbb{P}\in\mathcal{M}} \mathbb{E}_{\mathbb{P}}\left| \sup_{\mathbb{Q}\in\mathcal{M}^{[\omega]_{\mathcal{G}}}} \mathbb{E}_{\mathbb{Q}}(\xi) - \sup_{\mathbb{Q}\in\mathcal{M}^{[\omega]_{\mathcal{H}}}} \mathbb{E}_{\mathbb{Q}}(\xi)\right|$$