Additional information and pricing-hedging duality in robust framework

Anna Aksamit

based on joint work with
Zhaoxu Hou and Jan Obłoj

London - Paris Bachelier Workshop on Mathematical Finance
Pricing and hedging problems

How to determine the **prices** of exotic options?
How to hedge the positions in exotic options by using underlying assets and vanilla options?

- **Model-specific approach**: the price process of the underlying assets \((S_t)_{t \leq T}\) are modelled by some parametric family of stochastic processes.
- **Model-independent approach**: many possible models, weaker economic assumptions
  - Quasi-sure approach
  - Pathwise approach
Robust approach – an active field of research

Explicit bounds $LB \leq \mathcal{P}O_T \leq UB$ and robust super-/sub- hedges

Arbitrage considerations and robust FTAP

Pricing-hedging duality

Pathspace restrictions $A \not\subset \Omega$

Additional information

- $F$ regular agent/ common knowledge/ public information
- $G \supset F$ informed agent with additional information represented by the entire filtration
- Hedging $\xi$: how much the additional information is worth?
  \[
  \text{Price } F(\xi) - \text{Price } G(\xi)
  \]
- Super-hedging price:
  \[
  \inf \{ x : \exists (x, \gamma)-\text{super-hedges } \xi \}
  \]
  vs market model price:
  \[
  \sup_{P \in M} \mathbb{E}_P(\xi)
  \]
- No (duality) gap between these two values
Set-up

- Zhaoxu Hou and Jan Obłój *On robust pricing-hedging duality in continuous time*. 
Traded assets and information

- Stock price $S$ is the canonical process on
  
  $$\Omega := \{\omega \in C([0, T], \mathbb{R}_+^d) : \omega(0) = 1\}$$

- $\mathcal{F}$ is the filtration generated by $S$, i.e., $\mathcal{F}_t := \sigma(S_s : s \leq t)$

- $X_0, X_1, \ldots, X_n$ statically traded options which have prices $\mathcal{P}(X_i)$ at time 0, $X_0 = 1$ and $\mathcal{P}(X_0) = 1$

- $\mathcal{G}$ is the enlarged filtration $\mathcal{G}_t := \mathcal{F}_t \vee \mathcal{H}_t$, where $\mathcal{H} := (\mathcal{H}_t)_{t \leq T}$ is another filtration
  $\mathcal{G}$ is called the initial enlargement of $\mathcal{F}$ with random variable $Z$ if $\mathcal{H}_t = \sigma(Z)$
Equivalence relation and atoms

- \((\Omega, \mathcal{F}_T)\) measurable space and \(\mathcal{G}\) sub-\(\sigma\)-field of \(\mathcal{F}_T\), \(\omega, \tilde{\omega} \in \Omega\)

  \(\omega\) and \(\tilde{\omega}\) are \(\mathcal{G}\)-equivalent, \(\omega \sim_{\mathcal{G}} \tilde{\omega}\), if \(1_{\mathcal{G}}(\omega) = 1_{\mathcal{G}}(\tilde{\omega})\) holds \(\forall \mathcal{G} \in \mathcal{G}\)

  Note that \(\omega \sim_{\mathcal{T}_t} \tilde{\omega} \iff \omega_u = \tilde{\omega}_u\) for each \(u \leq t\), and \(\omega \sim_{\sigma(Z)} \tilde{\omega} \iff Z(\omega) = Z(\tilde{\omega})\)

- \([\omega]_{\mathcal{G}}\) denotes the equivalence class, or atom, in \(\Omega\) where \(\omega\) belongs to:

  \[ [\omega]_{\mathcal{G}} = \bigcap\{A : A \in \mathcal{G}, \omega \in A\} \]

- \((\Omega, \mathcal{G})\) is countably generated if there exists a sequence \((B_n)_{n \in \mathbb{N}} \subseteq \mathcal{G}\) such that \(\sigma((B_n)_{n \in \mathbb{N}}) = \mathcal{G}\). In this case each atom is \(\mathcal{G}\)-measurable
Trading strategies

- Integral of \( g : [0, T] \to \mathbb{R}^d \) of finite variation w.r.t. \( \omega \in \Omega \):
  \[
  \int_0^t g(u) d\omega(u) := g(t)\omega(t) - g(0)\omega(0) - \int_0^t \omega(u) dg(u)
  \]

- \( \gamma : \Omega \to D([0, T], \mathbb{R}^d) \) is \( \mathcal{G} \)-adapted if \( \gamma_t \) is \( \mathcal{G}_t \)-measurable, i.e. if
  \[
  \omega \sim_{\mathcal{G}_t} \bar{\omega} \quad \text{implies} \quad \gamma(\omega)_t = \gamma(\bar{\omega})_t
  \]
  and it is \( \mathcal{G} \)-admissible strategy if moreover it has finite variation and
  \[
  \int_0^t \gamma(\omega)_u dS_u(\omega) \geq -M(\omega) \quad \forall \omega, t \quad \text{for some} \ M \in L^0(\Omega, \mathcal{G}_0)
  \]

- A \( \mathcal{G} \)-admissible semi-static strategy is a pair \((X, \gamma)\) where
  \( X = A_0 + \sum_{i=1}^n A_i X_i \) for some \( \mathcal{G}_0 \)-measurable random variables \( A_i \) and \( \mathcal{G} \)-admissible strategy \( \gamma \).

  Initial cost of such a strategy is \( \mathcal{P}(X) = A_0 + \sum_{i=1}^m A_i \mathcal{P}(X_i) \).
  The set of all \( \mathcal{G} \)-admissible semi-static strategies is denoted by \( \mathcal{A}(\mathcal{G}) \).
The super-hedging price

\( \mathbb{G} \)-super-hedging price of \( \xi \) on \( A \in \mathcal{F}_T \):

\[
V^\mathbb{G}_A(\xi)(\omega) := \inf \{ \mathcal{P}(X)(\omega) : \exists (X, \gamma) \in \mathcal{A}(\mathbb{G}) \text{ such that } X(\tilde{\omega}) + \int_0^T \gamma(\tilde{\omega})_u dS_u(\tilde{\omega}) \geq \xi(\tilde{\omega}) \text{ for all } \tilde{\omega} \in A \}
\]

**Proposition**

The \( \mathbb{G} \)-super-hedging price on \( \Omega \) is constant on each \([\omega]\) and given by

\[
V^\mathbb{G}_\Omega(\xi)(\omega) = V^\mathbb{G}_{[\omega]}(\xi)
\]

where \([\omega]\) denotes the \( \mathcal{G}_0 \)-equivalence class containing \( \omega \).

It holds that

\[
V^\mathbb{G}_\Omega(\xi) \leq V^\mathbb{F}_\Omega(\xi)
\]
The market model price

- The set of $\mathcal{G}$-calibrated martingale measures concentrated on $A \in \mathcal{F}_T$:

$$\mathcal{M}_A^\mathcal{G} := \{ \mathbb{P} : S \text{ is a } (\mathbb{P}, \mathcal{G}) \text{-martingale}, \mathbb{P}(A) = 1 \\
\text{and } \mathbb{E}_\mathbb{P}(X_i|\mathcal{G}_0) = \mathbb{P}(X_i) \text{ for all } i \in \{1, \ldots, n\} \ \mathbb{P}\text{-a.s.} \}$$

- $\mathcal{G}$-market price of $\xi$ on $A \in \mathcal{F}_T$:

$$P_A^\mathcal{G}(\xi)(\omega) := \sup_{\mathbb{P} \in \mathcal{M}_A^\mathcal{G}} \mathbb{E}_\mathbb{P}(\xi|\mathcal{G}_0)(\omega)$$

Proposition

Assume each element of $\mathcal{G}$ is countably generated. Let $\mathbb{P} \in \mathcal{M}_\mathcal{X},\mathcal{P},\mathcal{G}$. Then, there exists a set $\Omega^\mathbb{P} \in \mathcal{G}_0$ with $\mathbb{P}(\Omega^\mathbb{P}) = 1$ and a version $\{\mathbb{P}_\omega\}$ of the regular conditional probabilities of $\mathbb{P}$ with respect to $\mathcal{G}_0$ such that for each $\omega \in \Omega^\mathbb{P}$, $\mathbb{P}_\omega \in \mathcal{M}_{\mathcal{X}},\mathcal{P},[\omega]_{\mathcal{G}_0}$.
The market model price

Let $A \in \mathcal{F}_T$. The $\mathcal{G}$-market price of $\xi$ on $A$ is defined by

$$P^G_A(\xi)(\omega) := \sup_{P \in \mathcal{M}_{\mathcal{X},P,A}^G} \bar{E}_P(\xi), \quad \omega \in \Omega,$$

where $\bar{E}_P(\xi) = E_P(\xi)$ for $\omega \in \Omega^P$ and $\bar{E}_P(\xi) = -\infty$ for $\omega \in \Omega \setminus \Omega^P$.

**Proposition**

The $\mathcal{G}$-market price on $\Omega$ is constant on each $[\omega]$ and given by

$$P^G_\Omega(\xi)(\omega) = P^G_{[\omega]}(\xi)$$

where $[\omega]$ is $\mathcal{G}_0$-equivalent class containing $\omega$.

It holds that $P^G_\Omega(\xi) \leq P^F_\Omega(\xi)$.
Lemma

The $\mathcal{G}$-super-hedging price $V_\Omega^\mathcal{G}(\xi)$ and the $\mathcal{G}$-market model price $P_\Omega^\mathcal{G}(\xi)$ of $\xi$ on $\Omega$ satisfy

$$V_\Omega^\mathcal{G}(\xi)(\omega) \geq P_\Omega^\mathcal{G}(\xi)(\omega) \quad \forall \omega \in \Omega.$$ 

Proof: $\mathcal{G}$-super-replicating portfolio $(X, \gamma) \in \mathcal{A}^M(\mathcal{G})$ on $[\omega]_{\mathcal{G}_0}$ and measure $\mathbb{P} \in \mathcal{M}[\omega]_{\mathcal{G}_0}$.

$\{\mathbb{P}_\nu\}$ regular conditional probabilities of $\mathbb{P}$ with respect to $\mathcal{G}_0$.

Since $\mathbb{P}(M \equiv \text{const}) = 1$,

$$E_{\mathbb{P}}(\xi) \leq E_{\mathbb{P}} \left( X + \int_0^T \gamma_u dS_u \right) \leq E_{\mathbb{P}}(X).$$
Duality
Let $G = F \lor \sigma(Z)$

Additional information arrives entirely at time 0

Atoms of $G_0$ are simply atoms of $\sigma(Z)$

Each atom can be seen as path restriction since on each atom the filtration $G$ and $F$ coincide, i.e., for each $\omega$

$$\forall G \in G_t \quad \exists F \in F_t \quad \text{s.t.} \quad [\omega]_{G_0} \cap G = [\omega]_{G_0} \cap F$$
Duality for $\mathbb{G} = \mathbb{F} \vee \sigma(Z)$

**Theorem**

Let $\mathbb{G} = \mathbb{F} \vee \sigma(Z)$ and assume that for each value $c$ of $Z$ we have

$$P_{\{Z=c\}}^\mathbb{F}(\xi) = V_{\{Z=c\}}^\mathbb{F}(\xi)$$

for any bounded uniformly continuous $\xi$.

Suppose that assumptions of Theorem Hou Obłój are satisfied. Then, *duality in $\mathbb{G}$ holds*, i.e.,

$$V_\Omega^\mathbb{G}(\xi)(\omega) = P_\Omega^\mathbb{G}(\xi)(\omega)$$

for any bounded uniformly continuous $\xi$.

**Proof:** One can show that:

$$P_{\{Z=c\}}^\mathbb{F}(\xi) = P_{\{Z=c\}}^\mathbb{G}(\xi) \leq V_{\{Z=c\}}^\mathbb{G}(\xi) = V_{\{Z=c\}}^\mathbb{F}(\xi)$$
Approximation of $A$: $A^\varepsilon = \{\omega \in \Omega : \inf_{v \in A} ||\omega - v|| \leq \varepsilon\}.$

$$\tilde{V}_A^F(\xi) := \inf\{P(X) : \exists (X, \gamma) \in A(F), \varepsilon > 0 \text{ s.t. } X + \int_0^T \gamma_u dS_u \geq \xi \text{ on } A^\varepsilon\}$$

$$\tilde{P}_A(\xi) := \lim_{\varepsilon \downarrow 0} \sup_{\varepsilon > 0, P \in M_{A,\varepsilon}^F} E_P(\xi) \quad \text{where}$$

$$M_{A,\varepsilon}^F := \{P : S \text{ is a } (P, F)\text{-mart.}, P(A^\varepsilon) = 1 - \varepsilon \text{ and } |E_P(X_i) - P(X_i)| \leq \varepsilon \forall i\}$$

**Theorem (Hou & Obłój)**

Assume that all payoffs $X_i$ are bounded and uniformly continuous and that for all $\varepsilon > 0$ there exists $P \in M_{A,\varepsilon}^F$.

Then for any bounded uniformly continuous $\xi : \Omega \to \mathbb{R}$

$$\tilde{V}_A(\xi) = \tilde{P}_A(\xi).$$

This theorem implies that

$$P_A(\xi) \leq V_A(\xi) \leq \tilde{V}_A(\xi) = \tilde{P}_A(\xi)$$
Duality for $G = F \lor \sigma(Z)$

**Theorem**

Let $G = F \lor \sigma(Z)$ and assume that for each value $c$ of $Z$ we have

$$P^{F}_{\{Z=c\}}(\xi) = \tilde{P}^{F}_{\{Z=c\}}(\xi)$$

for any bounded uniformly continuous $\xi$.

Suppose that assumptions of Theorem Hou Obłój are satisfied.

Then, **duality in $G$ holds**, i.e.,

$$\forall_{\Omega}^{G}(\xi)(\omega) = P^{G}_{\omega}(\xi)(\omega)$$

for any bounded uniformly continuous $\xi$.

**Example:** Assume no options and $d = 1$. No duality gap in $G$ holds:

- $Z = \sup_{t \in [0, T]} |\ln S_t|$
- $Z = \mathbb{1}_{\{a < S_t < b \ \forall t \in [0, T]\}}$ where $a < 1 < b$
Dynamic programming principle
Dynamic approach

Additional information $\sigma(Z)$ is disclosed at time $T_1 \in (0, T)$:

$$G_t = \mathcal{F}_t \text{ for } t \in [0, T_1) \text{ and } G_t = \mathcal{F}_t \lor \sigma(Z) \text{ for } t \in [T_1, T]$$

Assume $Z$ satisfies

$$Z(\omega) = \begin{cases} \tilde{Z}\left(\frac{\omega|_{[T_1,T]}}{\omega_{T_1}}\right) & \text{if } \omega_{T_1} > 0 \\ 1 & \text{if } \omega_{T_1} = 0 \end{cases} \text{ for r.v. } \tilde{Z} \text{ on } \Omega|_{[T_1,T]}.$$  

This encodes the idea that the additional information concerns only the evolution of prices after time $T_1$ irrespectively of the prices before time $T_1$.

**Theorem**  
*Duality in $\mathbb{G}$ holds, i.e., $V_{\Omega}^{\mathbb{G}}(\xi) = P_{\Omega}^{\mathbb{G}}(\xi)$ holds for any bounded uniformly continuous $\xi$.***
Dynamic approach

- Firstly solve the problem for each atom of $G_{T_1}$ on $[T_1, T]$ separately by the same arguments as for $F \lor \sigma(Z)$
- Secondly aggregate atoms of $G_{T_1}$ into atoms of $F_{T_1}$
- Apply dynamic principle
- Thus the problem is now reduced to $[0, T_1]$ and trading w.r.t $F$

$$V^G_{\Omega, [0, T]}(\xi) = V^F_{\Omega, [0, T_1]}(V^G_{\Omega, [T_1, T]}(\xi)) = V^F_{\Omega, [0, T_1]}(P^G_{\Omega, [T_1, T]}(\xi)) = P^F_{\Omega, [0, T_1]}(P^G_{\Omega, [T_1, T]}(\xi)) = P^G_{\Omega, [0, T]}(\xi).$$
Dynamic programming principle

Proposition

Let $\xi$ be uniformly continuous.

(i) Define

$$V_{\Omega}^{G, [T_1, T]}(\xi)(\omega) := \inf\{x : \exists \gamma \in A(G) s.t. x + \int_{T_1}^{T} \gamma_u dS_u \geq \xi \text{ on } [\omega]_{T_1}\}.$$ 

Then, $V_{\Omega}^{G, [T_1, T]}(\xi)$ is u.c. and

$$V_{\Omega}^{G, [0, T]}(\xi) = V_{\Omega}^{F, [0, T_1]} \left( V_{\Omega}^{G, [T_1, T]}(\xi) \right)$$

(ii) Define $P_{\Omega}^{G, [T_1, T]}(\xi)(\omega) := \sup_{\mathcal{P} \in \mathcal{M}_{[\omega]_{T_1}}} \mathbb{E}_{\mathcal{P}}(\xi)$. Then, $P_{\Omega}^{G, [T_1, T]}(\xi)$ is u.c. and

$$P_{\Omega}^{G, [0, T]}(\xi) = P_{\Omega}^{F, [0, T_1]} \left( P_{\Omega}^{G, [T_1, T]}(\xi) \right)$$
Conclusions

- Formulation of the duality problem for a general filtration with possibly non-trivial initial $\sigma$-field
- Translating original problem to the path restriction language from Hou & Obłój in case of an initial enlargement
- Disclosure of an additional information after initial time and dynamic programming principle
Thank You!
Dynamic programming principle

The path modification mapping $\alpha^{v,\tilde{v}}$ by

$$
\alpha(\omega) := \begin{cases} 
v|_{[0,T_1]} \otimes \frac{v_{T_1}}{v_{T_1}} \omega|_{[T_1,T]} & \omega \in B^{\tilde{v}} \\
\tilde{v}|_{[0,T_1]} \otimes \frac{\tilde{v}_{T_1}}{v_{T_1}} \omega|_{[T_1,T]} & \omega \in B^v \\
\omega & \omega \notin B^v \cup B^{\tilde{v}}
\end{cases}
$$

If the strategy $\gamma$ super-replicates on $B^v$, the strategy $\frac{v_{T_1}}{v_{T_1}} \gamma \circ \alpha + \frac{1}{v_{T_1}} \mathbb{1}_{[T_1,\tilde{T}]}$ super-replicates on $B^{\tilde{v}}$ and $\tilde{\xi}(\tilde{v}) \leq \tilde{\xi}(v) + e_{\tilde{\xi}}(||v - \tilde{v}||)$.

If $P \in \mathcal{M}_{B^v}^{G,[T_1,T]}$ then $\bar{P} = P \circ \alpha \in \mathcal{M}_{B^v}^{G,[T_1,T]}$ and $|E_P(\xi) - E_{\bar{P}}(\xi)| \leq e_{\tilde{\xi}}(||v - \tilde{v}||)$
Quantification of the value of the information

Initial enlargement – distances between σ-fields $d(\mathcal{G}, \mathcal{H})$:

$$
\sup_{G \in \mathcal{G}} \inf_{H \in \mathcal{H}} \sup_{P \in \mathcal{M}} P(G \Delta H) \lor \sup_{H \in \mathcal{H}} \inf_{G \in \mathcal{G}} \sup_{P \in \mathcal{M}} P(G \Delta H)
$$

$$
\sup_{0 \leq \xi \leq 1} \inf_{P \in \mathcal{M}} \sup_{Q \in \mathcal{M}[\omega]_{\mathcal{G}}} E_P \left\| \sup_{Q \in \mathcal{M}[\omega]_{\mathcal{H}}} E_Q(\xi) - \sup_{Q \in \mathcal{M}[\omega]_{\mathcal{H}}} E_Q(\xi) \right\|
$$