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Funding inclusive valuation as modified option pricing

Cristin Buescu - King's College London

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Joint work with Damiano Brigo and Marek Rutkowski

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The Idea

- Let $\mathbb{F} = (\mathcal{F}_t)$ where $\mathcal{F}_t := \sigma(S_u, u \leq t)$ for a traded asset (stock).
- For fixed maturity T let X be an $\mathcal{F}_T$-measurable integrable rv. and let A be a **contract** (vulnerable call option) that costs P_0 at time 0 and has the payoff X at maturity time T
$$X = 1_{\{\tau > T\}}(S_T - K)^+$$
with default time τ being a positive rv on $(\Omega, \mathcal{F}, \mathbb{P})$.
- The **price** P_t, $t \in [0, T]$, of this contract in a market with
  i) funding costs: unsecured funding account with the interest rate f;
  ii) stock (the underlying asset of the contract);
  iii) repo agreement on the stock with the repo rate h (at most β%);
  iv) credit risk: zero-recovery defaultable bond with the rate of return r^C (one-to-one with CDS).

  is of Black-Scholes w. dividends type (modified option pricing)
  $\Rightarrow$ sensitivity analysis.

Agenda

1. Historical perspective
2. Pricing benchmark products with funding, repo and credit
   - Modified cash-flow approach: Black-Scholes w. dividends
   - Martingale method approach: Black-Scholes w. dividends
3. Sensitivity analysis via the "funding Greeks"
4. Conclusions

Timeline:

- credit risk not considered pre-crisis
- 2008: 8 major credit events in 1 month (Sep 7 to Oct 8)
- sudden divergence between OIS and LIBOR
  (sign of presence of credit and liquidity risk affecting notional)
- impact on valuation of contracts: adjustments

Adjustments

- CVA (counterparty default cost)
- DVA (own default benefit)
  - benefit goes to creditors, not shareholders; discard it? (Albanese, Caenazzo, Crepey 2016)
  - can book large profit (Citigroup \$2.5 bn in 2009)
  - difficult to hedge (via correlated proxies)
- FVA (cost of funding the trade: treasury, creditors)
  - large! (\$1.5 bn JPM in 2014)
- KVA (cost of capital used)
  - no commonly agreed definition
  - not aware of amounts yet

Context

These contracts are OTC

- adjustments can provide a starting negotiating position
- move them to CCP? (challenge: reduce systemic risk while keeping trades feasible)
- market size still very large: $24.7 tn of derivatives (2012), $632 tn notional (BIS 2013)
- generally move towards simpler standardized contracts, shying away from features that may blow up

Issue

- Valuation becomes a nonlinear recursive problem
- Solution involves semi-linear PDEs and BSDEs (see the works of El Karoui, Peng, Quenez or Crepey)
- Get ad-hoc numerical solutions that are time-consuming and difficult to work with

Goal: "Our life is frittered away by detail... simplify, simplify."

Henry David Thoreau

- standard benchmark product (vulnerable call option)
- account for adjustments in price by including corresponding costs and benefits in the cash flow (approach of Brigo, Pallavicini 2014)
- derive and solve pricing equation via modified option pricing
- reconcile it with martingale measure approach (Bielecki, Jeanblanc, Rutkowski 2005)
- use it for sensitivity analysis: funding rate, repo, credit spread

Adjusted cash flow approach (Brigo, Pallavicini 2014)

Their pricing equation (11) becomes for our benchmark product

$$V_t = E^h\left[1_{\{\tau>T\}} D(t,T;f)(S_T - K)^+ \,|\, \mathcal{G}_t\right] \tag{1}$$

where $\mathbb{Q}^h$ is s.t. the drift of the risky asset is h:

$$dS_t = hS_t\,dt + \sigma S_t\,dW_t^h$$

$\mathbb{G} = (\mathcal{G}_t)$ is the full filtration including default information and the discount factor is

$$D(s,t;f) := \exp\left(-\int_s^t f_u\,du\right).$$

Assume a constant treasury rate f and use the counterparty pre-default intensity λ under $\mathbb{Q}^h$ defined in their equation (40)

$$1_{\{\tau>t\}}\lambda\,dt := \mathbb{Q}^h(\tau \in dt \,|\, \tau > t, \mathcal{F}_t),$$

to obtain the survival probability $G_t^h := \mathbb{Q}^h(\tau > t \,|\, \mathcal{F}_t) = e^{-\lambda t}$.

Change of filtration formula (Cor. 3.1.1 in Bielecki, Jeanblanc, Rutkowski 2004) gives:

$$V_t = 1_{\{\tau>t\}}(G_t^h)^{-1}\,\mathbb{E}^h[D(t,T;f)(S_T-K)^+G_T^h\,|\,\mathcal{F}_t].$$

If $\widetilde{V}$ denotes the $\mathbb{F}$-adapted pre-default price process s.t. $\forall t\in[0,T]$

$$1_{\{\tau>t\}}V_t = 1_{\{\tau>t\}}\widetilde{V}_t,$$

then

$$\begin{aligned}\widetilde{V}_t &= e^{-(\lambda+f)(T-t)}\,\mathbb{E}^h[(S_T-K)^+\,|\,\mathcal{F}_t]\\ &= e^{-(\lambda+f-h)(T-t)}\,\mathbb{E}^h[e^{-h(T-t)}(S_T-K)^+\,|\,\mathcal{F}_t]\\ &= e^{-(\lambda+f-h)(T-t)}\big(S_tN(d_1)-Ke^{-h(T-t)}N(d_2)\big). \qquad (2)\end{aligned}$$

Black-Scholes with dividend rate $\lambda+f-h$ and discount $\lambda+f$.

Main result (adjusted cash flow)

The valuation of a zero-recovery vulnerable call option in the presence of funding costs and repo can be mapped to the Black-Scholes formula with dividends

$$V_t = 1_{\{\tau > t\}} \Big(S_t e^{-(\lambda + f - h)(T-t)} N(d_1) - K e^{-(\lambda + f)(T-t)} N(d_2) \Big). \qquad (3)$$

Martingale method approach

Replication

- Let $\mathbb{F} = (\mathcal{F}_t)$ where $\mathcal{F}_t := \sigma(S_u, u \leq t)$ for a traded asset (stock).
- For fixed maturity date T let X be an $\mathcal{F}_T$-measurable integrable random variable.
- Assume that the default time τ is a positive random variable on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. It generates a filtration $\mathbb{H} = (\mathcal{H}_t)$ where $\mathcal{H}_t := \sigma(1_{\{\tau \leq u\}}, u \leq t)$, which is used to progressively enlarge $\mathbb{F}$ in order to obtain the full filtration $\mathbb{G} = (\mathcal{G}_t)$ with $\mathcal{G}_t := \mathcal{F}_t \vee \mathcal{H}_t$.
- Assume that $F_t := \mathbb{P}(\tau \leq t \,|\, \mathcal{F}_t)$ is a continuous, increasing function and $F_t < 1$ for any t (see Elliott, Jeanblanc, Yor (2000) in conjunction with the hypothesis (H) of Kusuoka)

Replication of a defaultable bond using CDS

Motivation for the assumption on the distribution of τ

- Replicate a zero-recovery defaultable bond maturing at T with a funding account (f_t) and a CDS on the bond issuer (spread r^{CDS}).
- The price process B in terms of the point process J (jumps to one at default) and of the pre-default price $\widetilde{B}$ is

$$B_t = 1_{\{J_t=0\}}\widetilde{B}_t = 1_{\{\tau>t\}}\widetilde{B}_t$$

- Assume that if no default occurred before t, then between t and $t+dt$ default may happen with a positive probability for arbitrarily small dt.

At time $t < \tau \wedge T$:

1. borrow $\widetilde{B}_t$ from the treasury and buy one defaultable bond;
2. buy a number $\widetilde{B}_t$ of CDS contracts on the same name.

At time $t + dt$:

3. if there is a default (i.e., $J_{t+dt} = 1$), then each of the $\widetilde{B}_t$ CDS contracts pays 1;
4. if there is no default (i.e., $J_{t+dt} = 0$), then sell the bond for $\widetilde{B}_{t+dt}$;
5. either way, pay the premium leg $r^{CDS}dt$ for each of the $\widetilde{B}_t$ CDS contracts and pay back the loan to the treasury: $\widetilde{B}_t(1 + f_t\,dt)$.

The overall gain over the time interval $(t, t + dt)$ is

$$\widetilde{B}_t 1_{\{J_{t+dt}=1\}} + \widetilde{B}_{t+dt} 1_{\{J_{t+dt}=0\}} - \widetilde{B}_t r^{CDS} dt - \widetilde{B}_t(1 + f_t\,dt).$$

Equating this to zero to ensure replication gives dynamics:

$$dB_t - B_t(r^{CDS} + f_t)\,dt + B_t\,dJ_t = 0, \tag{4}$$

and thus, since $B_T = 1_{\{\tau>T\}}$, for all $t \in [0, T]$

$$B_t = 1_{\{\tau>t\}} e^{-\int_t^T (r^{CDS}+f_u)\,du}. \tag{5}$$

Proposition

The above replication of the defaultable bond holds whenever the probability distribution of τ is continuous and its support includes $[0, T]$.

- This motivates our assumption on the distribution of the default time, and renders the replication independent of the particular distribution in this class.
- The replication should not be postulated a priori without this assumption.
- For instance, it fails when $\mathbb{P}(\tau \in (t_1, t_2)) = 0$ (see the analysis of Rutkowski (1999) for the discontinuous case).

Alternative derivation of bond pricing

- The replication can also be derived by martingale methods, taking any measure Q equivalent to P as postulated martingale measure.
- Key step: for any positive density on [0,T] $\exists!$ measure Q equivalent to P s.t. the distribution of τ under Q is exp(λ)
- It can be shown that Q is unique on $\mathcal{H}_t$ (information on defaults), so the model is complete and FTAP yields:

$$B_t = 1_{\{\tau > t\}} \, e^{-(\kappa + f)(T-t)}$$

(it can be shown that κ equals the CDS spread r^{CDS})

Let A be a **contract** (vulnerable call option) that costs P_0 at time 0 and has the payoff X at maturity time T

$$X = 1_{\{\tau > T\}}(S_T - K)^+.$$

Want the **price** P_t, $t \in [0, T]$, of this contract for an investor who replicates a long position using available financial instruments. The market has **primary assets** (A^1, A^2, A^3, A^4):

i) unsecured funding account with the interest rate f;
ii) stock (the underlying asset of the contract);
iii) repo agreement on the stock with the repo rate h;
iv) zero-recovery defaultable bond with the rate of return r^C issued by the counterparty.

(Recall one-to-one correspondence between r^C and r^{CDS} if needed.) At time t, the price P_t^i of the asset A^i is given by

$$P_t^1 = B_t^f, \;\; P_t^2 = S_t, \;\; P_t^3 = 0, \;\; P_t^4 = B_t$$

A **trading strategy** $\varphi = (\varphi^1, \varphi^2, \varphi^3, \varphi^4)$ gives the number of units of each primary asset purchased to build a portfolio.
A trading strategy φ is **admissible** if the repo is used for a fraction β of the required amount of stock, and the rest is funded by treasury.
At time $t \in [0, T]$ the **wealth** corresponding to an admissible φ is

$$V_t^\varphi = \sum_{i=1}^{4} \varphi_t^i P_t^i$$

A strategy φ is **self-financing** if for all $t \in [0, T]$

$$V_t^\varphi = V_0^\varphi + G_t^\varphi. \tag{6}$$

An admissible trading strategy φ **replicates** the payoff of a contract A if

$$V_T^\varphi = X.$$

The time t price of a contract A is the wealth V_t^φ

$$P_t := V_t^\varphi. \tag{7}$$

The existence of the specific primary assets in our market ensures that any claim is **attainable.**

The replicating strategy replicates not only the payoff of the option, but also the credit risk profile of a long position in the option.
The investor

1. buys $\beta\Delta_t$ repos, borrows $\beta\Delta_t S_t$ from treasury to buy and deliver $\beta\Delta_t$ shares, and receives $\beta\Delta_t S_t$ cash which is paid back to treasury;
2. borrows $(1-\beta)\Delta_t S_t$ from treasury and buys $(1-\beta)\Delta_t$ shares;
3. buys P_t/B_t units of the counterparty bond in order to match the value of this portfolio and the option payoff.

This portfolio corresponds to the following the admissible strategy

$$\theta_t := \left(-\frac{(1-\beta)\Delta_t S_t}{B_t^f}, (1-\beta)\Delta_t, \beta\Delta_t, \frac{P_t}{B_t} \right). \tag{8}$$

Standard replicating condition, self-financing and Ito lead to the pre-default pricing PDE for the function $v(t, s)$

$$v_t + \big((1-\beta)f + \beta h\big)s\,\frac{\partial v}{\partial s} + \frac{\sigma^2 s^2}{2}\frac{\partial v^2}{\partial s^2} - r_t^C v = 0 \tag{9}$$

with terminal condition $v(T, s) = (s - K)^+$.

Main result (replication)

The time t price of the vulnerable call option equals

$$P_t = 1_{\{\tau > t\}}\Big(S_t e^{-q(T-t)}N(d_1^q) - Ke^{-r^C(T-t)}N(d_2^q)\Big) \tag{10}$$

with $q = r^C - f^\beta$ and the **effective funding rate** defined as the weighted average: $f^\beta := (1-\beta)f + \beta h$.

For $\beta = 1$ (repo only) this is the same as for adjusted cash flow.

Martingale method

- The same pricing mapping into Black-Scholes w. dividends can be derived without resorting to PDEs
- Once we have the wealth dynamics use $\mathbb{Q}^\beta$ the probability measure equivalent to $\mathbb{P}$ s.t. the drift of S_t under $\mathbb{Q}^\beta$ is the effective funding rate f^β
- This yields a probabilistic representation of the price as a discounted expectation of the payoff under $\mathbb{Q}^\beta$ and Black-Scholes formula again

Consistency

- Valuation of contracts under funding, repo and credit by
  - adjusted cash flow approach
  - martingale measure approach

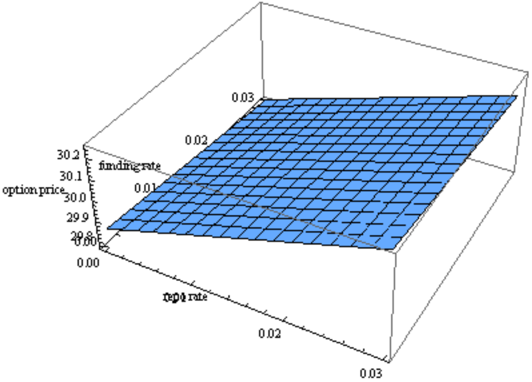
  maps into Black-Scholes formula w. dividends.
- Explicit formula allows for sensitivity analysis

$$P_t = 1_{\{\tau > t\}} \left(S_t e^{-q(T-t)} N(d_1^q) - K e^{-r^C (T-t)} N(d_2^q) \right)$$

  with $q = r^C - f^\beta$, $r^C = \lambda + f$, $f^\beta := (1-\beta)f + \beta h$, $\lambda = r^{CDS}$.
- Q: Which has the most price impact: funding, repo, or credit?

Numerical example

For $S_t = 80$, $K = 100$, $\sigma = 0.3$, $T - t = 0.1$, $r^{CDS} = 0.05$ the pre-default price of the vulnerable call is decreasing in the funding rate f
increasing in the repo rate h.



Introduce and evaluate "funding Greeks":

$$\begin{aligned} \partial_f &< 0 \ (f \ \text{ treasury rate}) \\ \partial_h &> 0 \ (h \ \text{ repo rate}) \\ \partial_\lambda &> 0 \ (\lambda \ \text{ is CDS spread}) \end{aligned}$$

Compute relative sensitivities:

$$\begin{aligned} \frac{\partial_h V}{V} &> T - t && \text{(repo)} \\ \frac{\partial_f V}{V} &= -(T - t) && \text{(funding)} \\ \frac{\partial_\lambda V}{V} &= -(T - t) && \text{(credit)} \end{aligned}$$

Valuation impact: repo > funding = credit.

Intepretation of "funding Greeks"

- vulnerable call = hybrid product: call on stock + long on bond
- if repo is used ($0 < \beta \leq 1$) the price increases in h (repo rate) due to the cost of hedging the option
- dual impact of f:
  - borrow cash (V increases with f)
  - invest in bond with rate $f + r^{CDS}$ (V decreases with f)
- overall impact of f may be negative as in the example (clear in the case $\beta = 1$ with no borrowing)

Conclusions

- Funding is not just a spread, but a complex nonlinear and recursive pricing problem (see expression below to appreciate what our simplification achieved)

$$\begin{aligned} P_t &= \int_t^T E^h[1_{\{u<\tau\}} D(t,u;f)(\Pi(u,u+du) + 1_{\{\tau \in du\}}\theta_u)|\mathcal{G}_t] \\ &+ \int_t^T E^h[1_{\{u<\tau\}} D(t,u;f)((f_u - c_u)M_u + (f_u^{N^C} - c_u)N_u^C + (f_u^{N^I} - c_u)N_u^I)|\mathcal{G}_t]du \end{aligned}$$

- Two alternative pricing approaches lead to the same result for a benchmark product when including funding, credit and repo
- Valuation (with adjustments) is mapped into Black-Scholes formula with dividends
- This allows for sensitivity analysis
- Pricing impact of repo rate is larger than that of funding or credit

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