Funding inclusive valuation as modified option pricing

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The Idea

- Let \( F = (\mathcal{F}_t) \) where \( \mathcal{F}_t := \sigma(S_u, u \leq t) \) for a traded asset (stock).
- For fixed maturity \( T \) let \( X \) be an \( \mathcal{F}_T \)-measurable integrable rv. and let \( A \) be a contract (vulnerable call option) that costs \( P_0 \) at time 0 and has the payoff \( X \) at maturity time \( T \)
  \[
  X = 1_{\{\tau > T\}} (S_T - K)^+ 
  \]
  with default time \( \tau \) being a positive rv on \( (\Omega, \mathcal{F}, \mathbb{P}) \).
- The price \( P_t, t \in [0, T] \), of this contract in a market with
  i) funding costs: unsecured funding account with the interest rate \( f \);
  ii) stock (the underlying asset of the contract);
  iii) repo agreement on the stock with the repo rate \( h \) (at most \( \beta \% \));
  iv) credit risk: zero-recovery defaultable bond with the rate of return \( r^C \)
    (one-to-one with CDS).

is of Black-Scholes w. dividends type (modified option pricing)
⇒ sensitivity analysis.
Agenda

1. Historical perspective

2. Pricing benchmark products with funding, repo and credit
   - Modified cash-flow approach: Black-Scholes w. dividends
   - Martingale method approach: Black-Scholes w. dividends

3. Sensitivity analysis via the "funding Greeks"

4. Conclusions
Historical perspective

Timeline:
- credit risk not considered pre-crisis
- 2008: 8 major credit events in 1 month (Sep 7 to Oct 8)
- sudden divergence between OIS and LIBOR
  (sign of presence of credit and liquidity risk affecting notional)
- impact on valuation of contracts: adjustments

Adjustments
- CVA (counterparty default cost)
- DVA (own default benefit)
  - benefit goes to creditors, not shareholders; discard it? (Albanese, Caenazzo, Crepey 2016)
  - can book large profit (Citigroup $2.5 bn in 2009)
  - difficult to hedge (via correlated proxies)
- FVA (cost of funding the trade: treasury, creditors)
  - large! ($1.5 bn JPM in 2014)
- KVA (cost of capital used)
  - no commonly agreed definition
  - not aware of amounts yet
Context

These contracts are OTC
- adjustments can provide a starting negotiating position
- move them to CCP? (challenge: reduce systemic risk while keeping trades feasible)
- market size still very large: $24.7 tn of derivatives (2012), $632 tn notional (BIS 2013)
- generally move towards simpler standardized contracts, shying away from features that may blow up

Issue
- Valuation becomes a nonlinear recursive problem
- Solution involves semi-linear PDEs and BSDEs (see the works of El Karoui, Peng, Quenez or Crepey)
- Get ad-hoc numerical solutions that are time-consuming and difficult to work with
Goal: "Our life is frittered away by detail... simplify, simplify."

Henry David Thoreau

- standard benchmark product (vulnerable call option)
- account for adjustments in price by including corresponding costs and benefits in the cash flow (approach of Brigo, Pallavicini 2014)
- derive and solve pricing equation via modified option pricing
- reconcile it with martingale measure approach (Bielecki, Jeanblanc, Rutkowski 2005)
- use it for sensitivity analysis: funding rate, repo, credit spread
Adjusted cash flow approach (Brigo, Pallavicini 2014)

Their pricing equation (11) becomes for our benchmark product

\[ V_t = E^h \left[ 1_{\{\tau > T\}} D(t, T; f)(S_T - K)^+ \mid \mathcal{G}_t \right] \]  \hspace{1cm} (1)

where \( Q^h \) is s.t. the drift of the risky asset is \( h \):

\[ dS_t = hS_t \, dt + \sigma S_t \, dW^h_t \]

\( \mathcal{G} = (\mathcal{G}_t) \) is the full filtration including default information and the discount factor is

\[ D(s, t; f) := \exp \left( - \int_s^t f_u \, du \right). \]

Assume a constant treasury rate \( f \) and use the counterparty pre-default intensity \( \lambda \) under \( Q^h \) defined in their equation (40)

\[ 1_{\{\tau > t\}} \lambda \, dt := Q^h(\tau \in dt \mid \tau > t, \mathcal{F}_t), \]

to obtain the survival probability \( G^h_t := Q^h(\tau > t \mid \mathcal{F}_t) = e^{-\lambda t} \).
Change of filtration formula (Cor. 3.1.1 in Bielecki, Jeanblanc, Rutkowski 2004) gives:

\[ V_t = 1_{\{\tau > t\}}(G_t^h)^{-1} \mathbb{E}^h[D(t, T; f)(S_T - K)^+ G_T^h | \mathcal{F}_t]. \]

If \( \tilde{V} \) denotes the \( \mathbb{F} \)-adapted pre-default price process s.t. \( \forall t \in [0, T] \)

\[ 1_{\{\tau > t\}} V_t = 1_{\{\tau > t\}} \tilde{V}_t, \]

then

\[ \tilde{V}_t = e^{-(\lambda + f)(T-t)} \mathbb{E}^h[(S_T - K)^+ | \mathcal{F}_t] \]

\[ = e^{-(\lambda + f - h)(T-t)} \mathbb{E}^h[e^{-h(T-t)}(S_T - K)^+ | \mathcal{F}_t] \]

\[ = e^{-(\lambda + f - h)(T-t)} (S_t N(d_1) - Ke^{-h(T-t)} N(d_2)). \quad (2) \]

Black-Scholes with dividend rate \( \lambda + f - h \) and discount \( \lambda + f \).
Main result (adjusted cash flow)

The valuation of a zero-recovery vulnerable call option in the presence of funding costs and repo can be mapped to the Black-Scholes formula with dividends

\[ V_t = 1_{\{\tau > t\}} \left( S_t e^{-(\lambda + f - h)(T-t)} N(d_1) - Ke^{-(\lambda + f)(T-t)} N(d_2) \right). \]  

(3)
Martingale method approach

Replication

- Let $\mathbb{F} = (\mathcal{F}_t)$ where $\mathcal{F}_t := \sigma(S_u, u \leq t)$ for a traded asset (stock).
- For fixed maturity date $T$ let $X$ be an $\mathcal{F}_T$-measurable integrable random variable.
- Assume that the default time $\tau$ is a positive random variable on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. It generates a filtration $\mathbb{H} = (\mathcal{H}_t)$ where $\mathcal{H}_t := \sigma(1_{\{\tau \leq u\}}, u \leq t)$, which is used to progressively enlarge $\mathbb{F}$ in order to obtain the full filtration $\mathbb{G} = (\mathcal{G}_t)$ with $\mathcal{G}_t := \mathcal{F}_t \lor \mathcal{H}_t$.
- Assume that $F_t := \mathbb{P}(\tau \leq t | \mathcal{F}_t)$ is a continuous, increasing function and $F_t < 1$ for any $t$ (see Elliott, Jeanblanc, Yor (2000) in conjunction with the hypothesis (H) of Kusuoka).
Replication of a defaultable bond using CDS

Motivation for the assumption on the distribution of $\tau$

- Replicate a zero-recovery defaultable bond maturing at $T$ with a funding account ($f_t$) and a CDS on the bond issuer (spread $r^{CDS}$).
- The price process $B$ in terms of the point process $J$ (jumps to one at default) and of the pre-default price $\bar{B}$ is
  \[
  B_t = 1_{\{J_t = 0\}} \bar{B}_t = 1_{\{\tau > t\}} \bar{B}_t
  \]
- Assume that if no default occurred before $t$, then between $t$ and $t + dt$ default may happen with a positive probability for arbitrarily small $dt$. 
At time $t < \tau \wedge T$:

1. borrow $\widetilde{B}_t$ from the treasury and buy one defaultable bond;
2. buy a number $\widetilde{B}_t$ of CDS contracts on the same name.

At time $t + dt$:

3. if there is a default (i.e., $J_{t+dt} = 1$), then each of the $\widetilde{B}_t$ CDS contracts pays 1;
4. if there is no default (i.e., $J_{t+dt} = 0$), then sell the bond for $\widetilde{B}_{t+dt}$;
5. either way, pay the premium leg $r^{CDS} dt$ for each of the $\widetilde{B}_t$ CDS contracts and pay back the loan to the treasury: $\widetilde{B}_t(1 + f_t dt)$.

The overall gain over the time interval $(t, t + dt)$ is

$$\widetilde{B}_t 1_{\{J_{t+dt}=1\}} + \widetilde{B}_{t+dt} 1_{\{J_{t+dt}=0\}} - \widetilde{B}_t r^{CDS} dt - \widetilde{B}_t(1 + f_t dt).$$

Equating this to zero to ensure replication gives dynamics:

$$dB_t - B_t(r^{CDS} + f_t) dt + B_t dJ_t = 0,$$

and thus, since $B_T = 1_{\{\tau > T\}}$, for all $t \in [0, T]$

$$B_t = 1_{\{\tau > t\}} e^{-\int_t^T (r^{CDS} + f_u) du}.$$
Proposition

The above replication of the defaultable bond holds whenever the probability distribution of $\tau$ is continuous and its support includes $[0, T]$. 

- This motivates our assumption on the distribution of the default time, and renders the replication independent of the particular distribution in this class.
- The replication should not be postulated a priori without this assumption.
- For instance, it fails when $\mathbb{P}(\tau \in (t_1, t_2)) = 0$ (see the analysis of Rutkowski (1999) for the discontinuous case).
Alternative derivation of bond pricing

- The replication can also be derived by martingale methods, taking any measure $Q$ equivalent to $P$ as postulated martingale measure.
- Key step: for any positive density on $[0,T]$ there exists a measure $Q$ equivalent to $P$ s.t. the distribution of $\tau$ under $Q$ is $\exp(\lambda)$.
- It can be shown that $Q$ is unique on $\mathcal{H}_t$ (information on defaults), so the model is complete and FTAP yields:

$$B_t = 1_{\{\tau > t\}} e^{-(\kappa + f)(T-t)}$$

(it can be shown that $\kappa$ equals the CDS spread $r^{CDS}$)
Let $A$ be a contract (vulnerable call option) that costs $P_0$ at time 0 and has the payoff $X$ at maturity time $T$

$$X = 1_{\{\tau > T\}}(S_T - K)^+.$$  

Want the price $P_t$, $t \in [0, T]$, of this contract for an investor who replicates a long position using available financial instruments. The market has primary assets $(A^1, A^2, A^3, A^4)$:

i) unsecured funding account with the interest rate $f$;
ii) stock (the underlying asset of the contract);
iii) repo agreement on the stock with the repo rate $h$;
iv) zero-recovery defaultable bond with the rate of return $r^C$ issued by the counterparty.

(Recall one-to-one correspondence between $r^C$ and $r^{CDS}$ if needed.)

At time $t$, the price $P^i_t$ of the asset $A^i$ is given by

$$P^1_t = B^f_t, \quad P^2_t = S_t, \quad P^3_t = 0, \quad P^4_t = B_t$$
A trading strategy \( \varphi = (\varphi^1, \varphi^2, \varphi^3, \varphi^4) \) gives the number of units of each primary asset purchased to build a portfolio.

A trading strategy \( \varphi \) is admissible if the repo is used for a fraction \( \beta \) of the required amount of stock, and the rest is funded by treasury.

At time \( t \in [0, T] \) the wealth corresponding to an admissible \( \varphi \) is

\[
V_t^\varphi = \sum_{i=1}^{4} \varphi_t^i P_t^i
\]

A strategy \( \varphi \) is self-financing if for all \( t \in [0, T] \)

\[
V_t^\varphi = V_0^\varphi + G_t^\varphi.
\] (6)

An admissible trading strategy \( \varphi \) replicates the payoff of a contract \( A \) if

\[
V_T^\varphi = X.
\]

The time \( t \) price of a contract \( A \) is the wealth \( V_t^\varphi \)

\[
P_t := V_t^\varphi.
\] (7)

The existence of the specific primary assets in our market ensures that any claim is attainable.
The replicating strategy replicates not only the payoff of the option, but also the credit risk profile of a long position in the option. The investor

1. buys $\beta \Delta_t$ repos, borrows $\beta \Delta_t S_t$ from treasury to buy and deliver $\beta \Delta_t$ shares, and receives $\beta \Delta_t S_t$ cash which is paid back to treasury;

2. borrows $(1 - \beta) \Delta_t S_t$ from treasury and buys $(1 - \beta) \Delta_t$ shares;

3. buys $P_t/B_t$ units of the counterparty bond in order to match the value of this portfolio and the option payoff.

This portfolio corresponds to the following the admissible strategy

$$\theta_t := \left( - \frac{(1 - \beta) \Delta_t S_t}{B_t^f}, (1 - \beta) \Delta_t, \beta \Delta_t, \frac{P_t}{B_t} \right).$$

(8)
Standard replicating condition, self-financing and Ito lead to the pre-default pricing PDE for the function $v(t, s)$

$$v_t + ((1 - \beta)f + \beta h)s \frac{\partial v}{\partial s} + \frac{\sigma^2 s^2}{2} \frac{\partial^2 v}{\partial s^2} - r_t^C v = 0 \tag{9}$$

with terminal condition $v(T, s) = (s - K)^+$. 

Main result (replication)

The time $t$ price of the vulnerable call option equals

$$P_t = 1_{\{\tau > t\}} \left( S_t e^{-q(T-t)} N(d_1^q) - Ke^{-r_C(T-t)} N(d_2^q) \right) \tag{10}$$

with $q = r^C - f^\beta$ and the effective funding rate defined as the weighted average: $f^\beta := (1 - \beta)f + \beta h$.

For $\beta = 1$ (repo only) this is the same as for adjusted cash flow.
The same pricing mapping into Black-Scholes w. dividends can be derived without resorting to PDEs.

Once we have the wealth dynamics use $Q^\beta$ the probability measure equivalent to $\mathbb{P}$ s.t. the drift of $S_t$ under $Q^\beta$ is the effective funding rate $f^\beta$.

This yields a probabilistic representation of the price as a discounted expectation of the payoff under $Q^\beta$ and Black-Scholes formula again.
Consistency

Valuation of contracts under funding, repo and credit by
- adjusted cash flow approach
- martingale measure approach
maps into Black-Scholes formula w. dividends.

Explicit formula allows for sensitivity analysis

\[ P_t = 1_{\{\tau > t\}} \left( S_t e^{-q(T-t)} N(d_1^q) - Ke^{-r^C(T-t)} N(d_2^q) \right) \]

with \( q = r^C - f^\beta, \quad r^C = \lambda + f, \quad f^\beta := (1 - \beta)f + \beta h, \quad \lambda = r^{CDS} \).

Q: Which has the most price impact: funding, repo, or credit?
Numerical example

For $S_t = 80, K = 100, \sigma = 0.3, T - t = 0.1, r^{CDS} = 0.05$ the pre-default price of the vulnerable call is decreasing in the funding rate $f$ increasing in the repo rate $h$. 

Figure: Option price is increasing in repo rate $h$ and decreasing in funding rate $f$. 
Introduce and evaluate "funding Greeks":

\[ \partial_f < 0 \quad (f \text{ treasury rate}) \]
\[ \partial_h > 0 \quad (h \text{ repo rate}) \]
\[ \partial_\lambda > 0 \quad (\lambda \text{ is CDS spread}) \]

Compute relative sensitivities:

\[ \frac{\partial_h V}{V} > T - t \quad \text{(repo)} \]
\[ \frac{\partial_f V}{V} = -(T - t) \quad \text{(funding)} \]
\[ \frac{\partial_\lambda V}{V} = -(T - t) \quad \text{(credit)} \]

Valuation impact: repo > funding = credit.
Intepretation of "funding Greeks"

- vulnerable call = hybrid product: call on stock + long on bond
- if repo is used \((0 < \beta \leq 1)\) the price increases in \(h\) (repo rate) due to the cost of hedging the option
- dual impact of \(f\):
  - borrow cash \((V\) increases with \(f\))
  - invest in bond with rate \(f + r^{CDS}\) \((V\) decreases with \(f\))
- overall impact of \(f\) may be negative as in the example (clear in the case \(\beta = 1\) with no borrowing)
Conclusions

- Funding is not just a spread, but a complex nonlinear and recursive pricing problem (see expression below to appreciate what our simplification achieved)

\[
P_t = \int_t^T E^h[1\{u<\tau\} \cdot D(t, u; f)(\Pi(u, u + du) + 1\{\tau \in du\} \theta_u)]|G_t|
+ \int_t^T E^h[1\{u<\tau\} \cdot D(t, u; f)((f_u - c_u)M_u + (f_{u}^{NC} - c_u)N_u^C + (f_{u}^{NI} - c_u)N_u^I)|G_t]du
\]

- Two alternative pricing approaches lead to the same result for a benchmark product when including funding, credit and repo

- Valuation (with adjustments) is mapped into Black-Scholes formula with dividends

- This allows for sensitivity analysis

- Pricing impact of repo rate is larger than that of funding or credit
References


