Asymptotics and calibration for American options

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- Important volumes of American options traded in exchange markets
- Typically only American options (and no European option) are traded on individual stocks (as opposed to stock indexes)

Question

- Calibration of a stochastic model (say local volatility) to American options data ?

- The calibration of a model to American options is possible with pricing via (for example)
  - free-boundary PDE
  - tree methods

  $+$ a least-square optimization

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  - compute implied volatility from America options
  - input this smile *as if it was European* $\rightsquigarrow$ then calibrate a model with standard methods (ex. local volatility from Dupire's formula).
  - Issues : consistency, lack of a unique smile for American options,...

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- Refined asymptotics for the exercise boundary close to maturity
- Semi-closed approximation for the American put price *in terms of* a parametric European smile

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In this presentation :

- Refined asymptotics for the exercise boundary close to maturity
- Semi-closed approximation for the American put price *in terms of* a parametric European smile

- ▶ A calibration recipe for American options

Reminder : calibration of local volatility to European options

$$dX_t = rX_t dt + \sigma(t, X_t) dW_t$$

- $P(T,K) = \mathbb{E}[e^{-rT}(K - X_T)^+]$ denotes the Put price

  Dupire forward PDE :

$$\partial_T P(T,K) + rK\partial_K P(T,K) = \frac{1}{2}\sigma(T,K)^2 \partial_{KK} P(T,K)$$

  $\rightsquigarrow$ can reconstruct the local volatility function $\sigma(\cdot)$ from the observed put prices $P(\cdot)$: Dupire's formula

$$\sigma(T,K)^2 = \frac{\partial_T P(T,K) + rK\partial_K P(T,K)}{\frac{1}{2}\partial_{KK} P(T,K)}$$

Reminder : pricing of American options [1]

$$dX_t = rX_t dt + \sigma(X_t) dW_t$$

- American put : holder gets $(K - X_\tau)^+$ if exercised at $\tau \leq T$
- Valuation : several formulations

1. here without dividends

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  - Bensoussan 84, Karatzas 88 : optimal stopping problem

$$u(t,x) = \sup_{\tau \in [t,T]} \mathbb{E}\left[e^{-r(\tau - t)}(K - X_\tau^{t,x})^+\right]$$

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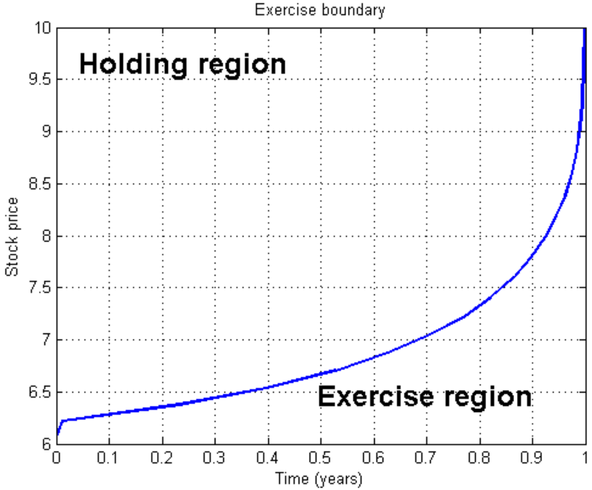
  - Free-boundary PDE (dates back to McKean 65)

$$\begin{cases} \partial_t u + rx\partial_x u + \frac{1}{2}\sigma(x)^2 \partial_{xx} u - ru = 0 & t < T, x > x(t) \\ u(t,x) = (K-x)^+ & t < T, x \le x(t) \\ u(t,x) > (K-x)^+ & t < T, x > x(t) \\ u(T,x) = (K-x)^+ & t = T, x > 0. \end{cases}$$

$x(t) = \inf\{x > 0 : u(t,x) > (K-x)^+\}$ is the exercise boundary.

1. here without dividends

American put : exercise boundary



source mathworks.com

Yet a reminder : exercise boundary & early premium

Early premium formula for the American Put

$$u(t,x) = \mathbb{E}\left[e^{-r(T-t)}(K - X_T^{t,x})^+\right] + rK\int_t^T \mathbb{E}\left[e^{-r(s-t)} 1_{X_s^{t,x} \le x(s)}\right] ds$$

$$= P_{t,x}(T,K) + rK\int_t^T e^{-r(s-t)} F_{t,x}(s, x(s)) ds$$

See McKean 65, Moerbeke 76, Myneni 92 (and many others)

Exercise boundary close to maturity

- τ : time to maturity.
- Parabolic behavior $\sqrt{\tau}$ with logarithmic correction :

$$x(T-\tau) \sim K - \sigma(K)\sqrt{\tau \ln\left(\frac{1}{\tau}\right)}, \qquad \text{as } \tau \to 0$$

Barles et al. 92, Lamberton 95, in the Black-Scholes model $\sigma(K) = \sigma K$; Chevalier 05 for general local volatility $\sigma(\cdot)$.

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- Refinements in the Black-Scholes model :

$$x(T-\tau) \sim K - \sigma K\sqrt{\tau \ln\left(\frac{\gamma_{BS}}{\tau}\right)}, \qquad \text{as } \tau \to 0$$

where $\gamma_{BS} = \frac{\sigma^2}{8\pi r^2}$. See Evans et al. 02

Exercise boundary close to maturity

Theorem : as $\tau \to 0$,

$$x(T-\tau) = K - \sigma(K)\sqrt{\tau\left[\ln\frac{\gamma}{\tau} + \phi(\tau)\right]}$$

where now $\phi(\tau) = -2\ln\left(1 + \frac{1}{\ln\frac{\tau}{\gamma}} + \left(\frac{1}{\ln\frac{\tau}{\gamma}}\right)^2 + O\left(\frac{1}{\ln\frac{\tau}{\gamma}}\right)^3\right), \quad \gamma = \frac{\sigma(K)^2}{8\pi(rK)^2}$

- Some comments :

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- Some comments :
  - γ is homogeneous to time. It plays the role of a time-scale. In BS model, $\gamma_{BS} = \frac{1}{8\pi}\left(\frac{\sigma}{r}\right)^2 \rightsquigarrow$ typical values $\gamma_{BS} \approx 4$.

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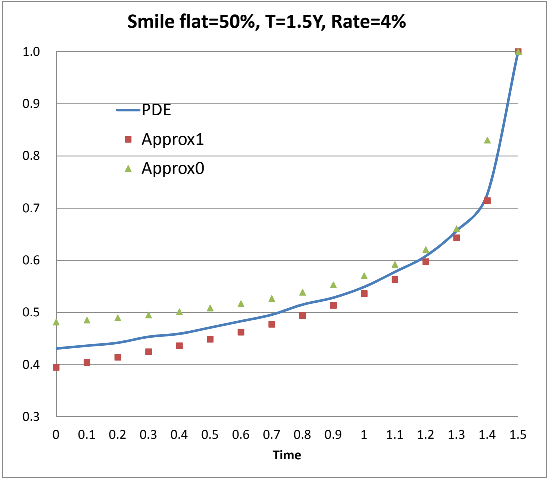
  - The blue term $[\cdot]$ above can (trivially !) be rewritten as

$$\ln\left(\frac{1}{\tau}\right) + \ln(\gamma) - 2\ln\left(1 + \frac{1}{\zeta} + \left(\frac{1}{\zeta}\right)^2 + O\left(\frac{1}{\zeta}\right)^3\right), \quad \zeta = \ln\frac{\tau}{\gamma}$$

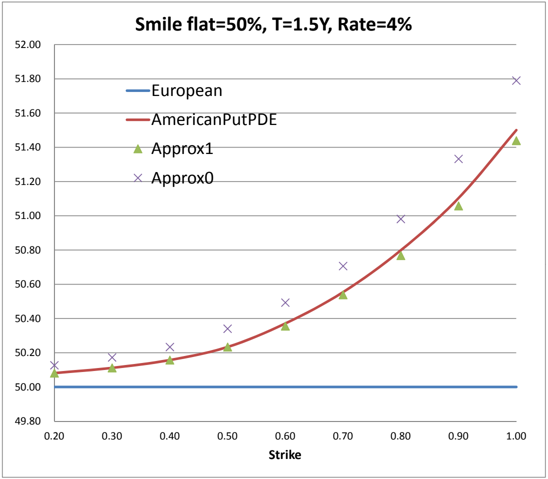
$\rightsquigarrow$ $\ln(\gamma)$ is an important correction in front of $\ln\left(\frac{1}{\tau}\right)$.

Exercise boundary (Black-Scholes model)



Exercise boundary in the BS model, $\sigma_{BS} = 0.5$, $r = 4\%$, $K = 1$, $T = 1.5Y$

American put price (Black-Scholes model)



American put price with flat BS smile $\sigma_{BS} = 0.5$

Extensions

- to **homogeneous stochastic volatility**

$$
\begin{aligned}
dX_t &= rX_t dt + a_t \sigma(X_t) dW_t \\
da_t &= b(a_t) dt + \nu(a_t) dZ_t
\end{aligned}
$$

See end of this presentation for an example

Extensions (II)

- to **inhomogeneous local volatility**

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and now $\gamma = \frac{\sigma(T,K)^2}{8\pi(rK)^2}$

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- Inclusion of dividend rate $d \neq 0$.

American options from Vanillas

From the early premium formula,

$$A(T,K) := u^{T,K}(0,x) \approx \mathrm{Put}(T,K) + rK \int_0^T \partial_K \mathrm{Put}(s, x_{approx}(s)) ds$$

- American options from Vanillas in a given local vol model
- up to a numerical quadrature for $\int_0^T ds$, using the explicit expression $x_{approx}(s)$

American options from Vanillas (reparameterisation)

- Recall Dupire's formula from a European implied vol $\sigma^{\mathrm{Eur}}(T,K)$:

$$\text{denote} \qquad \mathrm{Put}_{\mathrm{IV}}(T,K) = \mathrm{Put}_{\mathrm{BS}}(T,K;\sigma^{\mathrm{Eur}}(T,K))$$

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$$\sigma_{\mathrm{loc}}(T, K)^2 = \frac{\partial_T \mathrm{Put}_{\mathrm{IV}}(T, K) + rK \ \mathrm{Put}_{\mathrm{IV}}(T, K)}{\frac{1}{2} \partial_{KK} \mathrm{Put}_{\mathrm{IV}}(T, K)}$$

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&= \frac{(\sigma^{\mathrm{Eur}})^2 + 2T\,\sigma^{\mathrm{Eur}}\partial_T\sigma^{\mathrm{Eur}}}{T\,\sigma^{\mathrm{Eur}}(\sigma^{\mathrm{Eur}})'' - \frac{T^2(\sigma^{\mathrm{Eur}})^2}{4}((\sigma^{\mathrm{Eur}})')^2 + \left(1 - \frac{x(\sigma^{\mathrm{Eur}})'}{\sigma^{\mathrm{Eur}}}\right)^2}
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- Allows to write

$$x_{\mathrm{IV}}(T-\tau) = K - \sigma_{\mathrm{loc}}(T,K)\sqrt{\tau\left[\ln\frac{\gamma_{\mathrm{IV}}}{\tau} - 2\ln\left(1 + \frac{1}{\ln\frac{\tau}{\gamma_{\mathrm{IV}}}} + \left(\frac{1}{\ln\frac{\tau}{\gamma_{\mathrm{IV}}}}\right)^2\right)\right]}$$

where

$$\gamma_{\mathrm{IV}} = \frac{\sigma_{\mathrm{loc}}(T,K)^2}{8\pi r^2K^2} \quad \longleftarrow \text{ written in terms of } \sigma^{\mathrm{Eur}}(\cdot)$$

American options from Vanillas

Final product : American put in terms of a *parameterisation* of the European smile :

$$A(T,K) \approx \mathrm{Put}_{\mathrm{IV}}(T,K) + rK \int_0^T \partial_K \mathrm{Put}_{\mathrm{IV}}(s, x_{\mathrm{IV}}(s))ds$$

Calibration recipe :

American options from Vanillas

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- Start from a arbitrage–free parametric implied vol surface $\sigma^{\mathrm{Eur}}(\cdot)$
  - for example, SSVI (Gatheral and Jacquier 14)

Calibrate the (parametric !) r.h.s. above to American option data by least-square minimization.

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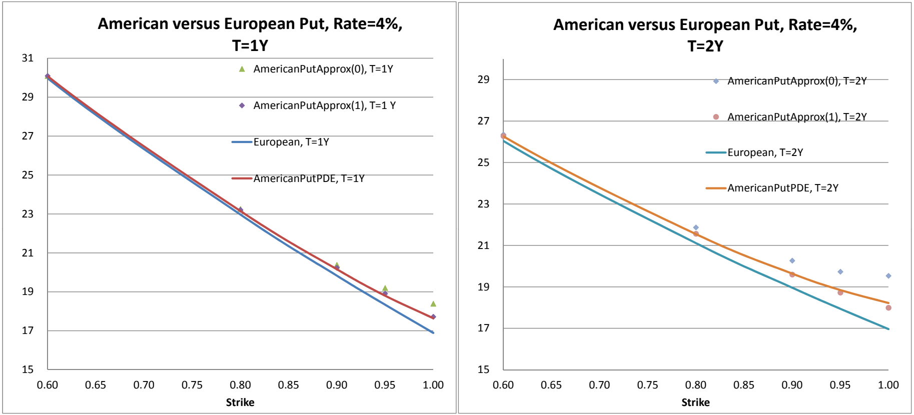
- *Advantage* : no PDE solver/Monte-Carlo. The only approximation is in the computation of $x_{\mathrm{IV}}(\cdot)$.

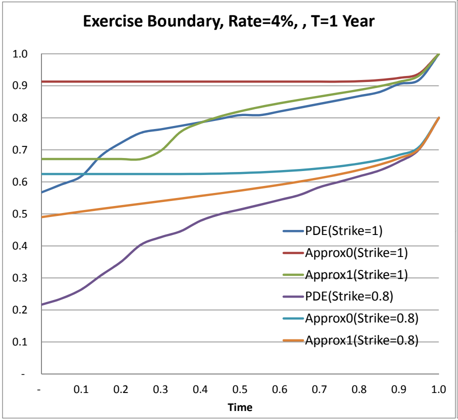
Thank you for your attention

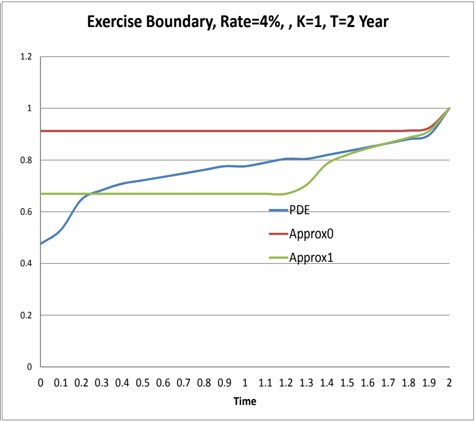
Local volatility : put price, two maturities



American put prices for $T = 1Y$ (left) and $T = 2Y$ (right), local vol

Local volatility : exercise boundary





Exercise boundary for $T = 1Y$ and different strikes (left) and $T = 2Y$ (right), local vol

Homogeneous stochastic volatility

$$dX_t = rX_t dt + \sigma(X_t) a_t dW_t$$
$$da_t = b(a_t)dt + \nu(a_t)dZ_t$$

Theorem

$$d(K, a_{\min}|x(\tau, a), a) \sim \sqrt{\tau \ln\left(\frac{1}{\tau}\frac{1}{8\pi(rK)^2}\left(d_x(K, a|K, a)C^2(K)a^2\right)^2\right)}, \quad \tau \to 0$$

where d is the geodesic distance associated to (X, a) and
$a_{\min} \equiv \mathrm{argmin}_{a'} d^2(K, a'|x(\tau, a), a)$.

Example : SABR model

SABR model ($\beta = 1$) corresponds $dX_t = rX_t dt + X_t a_t dW_t$, $da_t = \nu a_t dZ_t$, $d\langle W, Z\rangle_t = \rho dt$

The previous theorem gives :

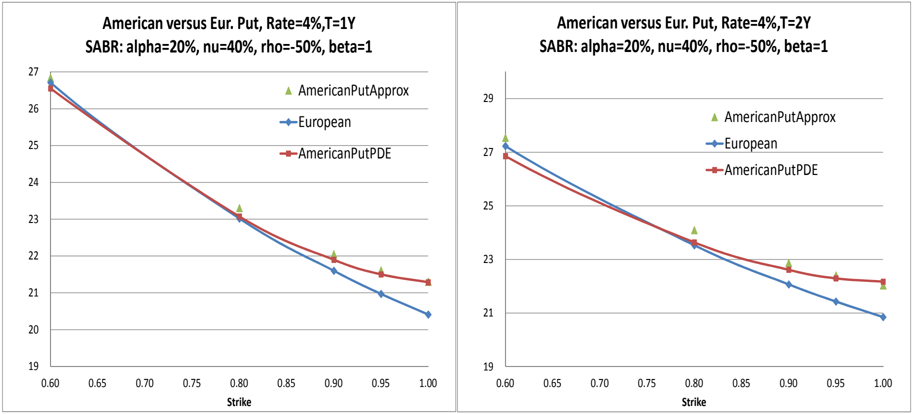
$$\frac{1}{\nu}\cosh^{-1}\left(\frac{-q\nu\rho - a\rho^2 + \sqrt{a^2 + 2\nu\rho\nu a q + \nu^2 q^2}}{a(1-\rho^2)}\right) = \sqrt{\tau \ln\left(\frac{\gamma}{\tau} a^2 \frac{1}{(1-\rho^2)}\right)\left(1 + O\left(\frac{1}{\ln \tau}\right)\right)}$$

$$\text{where } q = \int_{x(\tau,a)}^{K} \frac{dx}{\sigma(x)}$$

Corollary

$$x(\tau, a) = K - a\sigma(K)\sqrt{\tau \ln\left(\frac{\gamma}{\tau} a^2 \frac{1}{(1-\rho^2)}\right)}(1 + o(1)), \qquad \tau \to 0$$

where (again) $\gamma = \frac{\sigma(K)^2}{8\pi(rK)^2}$.



American put price for $T = 1Y$ (left) and $T = 2Y$ (right), SABR stoch vol model