Game options in an imperfect market with default

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Plan

- Game options: Literature/Contribution
- Framework
- Linear/Nonlinear pricing
- Nonlinear pricing of Game options
- Nonlinear pricing of Game options in the case with ambiguity.

Game options

Extend the setup of American options by allowing the seller to cancel the contract (introduced by Kifer in 2000).

- If the buyer exercises the contract at time τ, he gets ξ_τ from the seller
- If the seller cancels at σ before τ, then he has to pay ζ_σ to the buyer
- $\zeta_t - \xi_t \geq 0$, for all t represents the penalty for the seller for the cancellation of the contract.

The seller pays to the buyer the payoff $I(\tau, \sigma) := \xi_\tau \mathbf{1}_{\tau \leq \sigma} + \zeta_\sigma \mathbf{1}_{\tau > \sigma}$ at the terminal time $\tau \wedge \sigma$.

Literature

In a *perfect complete market*, Kifer (2000) shows both in the CRR discrete time-model and in the Black-Scholes model (with ξ and ζ continuous), that the superhedging price is equal to the value function of a Dynkin game:

$$u_0 = \sup_{\tau} \inf_{\sigma} \mathbf{E}_Q[\tilde{\xi}_\tau \mathbf{1}_{\tau \leq \sigma} + \tilde{\zeta}_\sigma \mathbf{1}_{\tau > \sigma}] = \inf_{\sigma} \sup_{\tau} \mathbf{E}_Q[\tilde{\xi}_\tau \mathbf{1}_{\tau \leq \sigma} + \tilde{\zeta}_\sigma \mathbf{1}_{\tau > \sigma}],$$

where $\tilde{\xi}_t$, $\tilde{\zeta}_t$ are the discounted values of ξ_t,ζ_t and $\mathbf{E}_Q$ represents the expectation under the unique martingale probability measure Q of the market model.

Literature

Other works: on pricing of games options or more sophisticated game-type financial contracts (e.g. swing game options)

- $\hookrightarrow$ In the discrete time: Dolinsky and Kifer (2007), Dolinsky and al. (2011)
- $\hookrightarrow$ In the continuous time perfect market model with continuous payoffs - Hamadène (2006), Kifer (2013)
- $\hookrightarrow$ Pricing of game options in a market with default - Bielecki and al. (2009)

Contribution

- Study the game options (pricing and superhedging) in **the case of imperfections in the market** taken into account via ***the nonlinearity*** *of the wealth dynamics* (in the case when there also exists the *possibility of a default* and **the payoffs are irregular**).
- Study game options under **model uncertainty**, in particular ambiguity on the default probability.

Model

Let $(\Omega, \mathcal{G}, \mathbb{P})$ be a complete probability space equipped with

- a unidimensional standard Brownian motion W
- a jump process N defined by $N_t = \mathbf{1}_{\vartheta \leq t}$ for any $t \in [0, T]$, where ϑ is a r.v. which modelizes a default time. We assume that this default can appear at any time that is $P(\vartheta > t) > 0$ for any $t \in [0, T]$.

We denote by $\mathbb{G} = \{\mathcal{G}_t, 0 \leq t \leq T\}$ the complete natural filtration of W and N. We suppose that W is a $\mathcal{G}$-brownian motion.
Let M be the compensated martingale of the process N:

$$M_t = N_t - \int_0^t \lambda_s ds \,.$$

The process (λ_t) is called *intensity*. λ vanishes after the default time ϑ.

Financial market

- 3 assets: prices S^0, S^1, S^2 satisfying:

$$\begin{cases} dS^0_t = S^0_t r_t dt \\ dS^1_t = S^1_t [\mu^1_t dt + \sigma^1_t dW_t] \\ dS^2_t = S^2_t [\mu^2_t dt + \sigma^2_t dW_t - dM_t]. \end{cases}$$

The price process S^2 admits a discontinuity at time ϑ. All processes $\sigma^1, \sigma^2, r, \mu^1, \mu^2$ are $\mathcal{G}$-predictable. We set $\sigma = (\sigma^1, \sigma^2)'$. We assume $\sigma^1, \sigma^2 > 0$, and the coefficients $\sigma^1, \sigma^2, \mu^1, \mu^2, (\sigma^1)^{-1}, (\sigma^2)^{-1}$ are bounded. The interest rate r is lower bounded.

Financial market

- Let us consider an investor, endowed with an initial wealth x.
  At t, he chooses the amount φ_t^1 (resp. φ_t^2) invested S^1 (resp S^2).
  φ^2 vanishes after ϑ.
  $\varphi_. = (\varphi_t^1, \varphi_t^2)'$ is called *risky assets stategy*.
  Let $V_t^{x,\varphi}$ (or V_t) $=$ value of the portfolio.

Linear pricing

Perfect market model

$$dV_t = (r_t V_t + \varphi^1_t \theta^1_t \sigma^1_t - \lambda_t \varphi^2_t \theta^2_t)dt + \varphi'_t \sigma_t dW_t - \varphi^2_t dM_t,$$

where $\theta^1_t := \dfrac{\mu^1_t - r_t}{\sigma^1_t}$ and $\theta^2_t := -\dfrac{\mu^2_t - \sigma^2_t \theta^1_t - r_t}{\lambda_t} \mathbf{1}_{t \leq \vartheta}$.

Consider an European option with maturity T and payoff ξ. The unique solution (X, Z, K) of the **linear BSDE** with default:

$$-dX_t = \underbrace{-(r_t X_t + (Z_t + \sigma^2_t K_t)\theta^1_t + K_t \theta^2_t \lambda_t)}_{g(t,y,z,k) = -(r_t y + (z + \sigma^2_t k \mathbf{1}_{t \leq \vartheta}))\theta^1_t + \theta^2_t \lambda_t k} dt - Z_t dW_t - K_t dM_t; \; X_T = \xi.$$

provides the replicating portfolio. The heging strategy φ is such that

$$\varphi'_t \sigma_t = Z_t; \;\; \varphi^2_t = -K_t.$$

This defines a change of variables $\Phi(Z, K) := (\varphi^1, \varphi^2)$.

Nonlinear pricing

The imperfect market model $\mathcal{M}^g$

The imperfections in the market are taken into account via the *nonlinearity* of the dynamics of the wealth $V_t^{x,\phi}$:

$$-dV_t = g(t, V_t, \varphi_t{}'\sigma_t, \varphi_t^2)dt + \varphi_t{}'\sigma_t dW_t - \varphi_t^2 dM_t,$$

Consider an European option with maturity $S \in [0, T]$ and terminal payoff ξ. The unique solution (X, Z, K) of the **nonlinear BSDE** with default

$$-dX_t = g(t, X_t, Z_t, K_t)dt - Z_t dW_t - K_t dM_t, \; X_S = \xi.$$

gives the hedging price (X) and the hedging strategy $(\varphi^1, \varphi^2) := \Phi(Z, K)$.
This leads to a nonlinear pricing system (introduced by El Karoui-Quenez), denoted by $\mathcal{E}^g : \forall S \in [0, T], \forall\ \xi \in L_2$

$$\mathcal{E}^g_{t,S}[\xi] := X_t(S, \xi), \quad t \in [0, S].$$

Nonlinear pricing

The imperfect market model $\mathcal{M}^g$

Examples of imperfections:

- Different borrowing and lending interest rates R_t and r_t with $R_t \geq r_t$.
  $g(t, V_t, \varphi_t \sigma_t, -\varphi_t^2) =$
  $-(r_t V_t + \varphi_t^1 \theta_t^1 \sigma_t^1 - \varphi_t^2 \lambda_t \theta_t^2) + (R_t - r_t)(V_t - \varphi_t^1 - \varphi_t^2)^-$
- Large investor whose trading strategy φ_t impacts the market prices: $r_t(\omega) = \bar{r}(t, \omega, \varphi_t)$ and similarly for $\sigma^1, \sigma^2, \theta^1, \theta^2$.

  $$g(t, V_t, \varphi_t \sigma_t, -\varphi_t^2) = -\bar{r}(t, \varphi_t) V_t - \varphi_t^1 (\bar{\theta}^1 \bar{\sigma}^1)(t, \varphi_t) + \varphi_t^2 \lambda_t \bar{\theta}^2(t, \varphi_t).$$

Nonlinear pricing and hedging

The imperfect market model $\mathcal{M}^g$

Definition [Driver, λ-*admissible* driver]

- A function g is said to be a *driver* if $g : [0,T] \times \Omega \times \mathbb{R}^3 \to \mathbb{R}$; $(\omega, t, y, z, k) \mapsto g(\omega, t, y, z, k)$ is $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^3)-$ measurable, and $g(., 0, 0, 0) \in \mathbb{H}_2$.
- A driver g is called a λ-*admissible driver* if moreover there exists a constant $C \geq 0$ such that $dP \otimes dt$-a.s. ,

for each (y, z, k), (y_1, z_1, k_1), (y_2, z_2, k_2),

$$|g(\omega, t, y, z_1, k_1) - g(\omega, t, y, z_2, k_2)| \leq C(|z_1 - z_2| + \sqrt{\lambda_t}|k_1 - k_2|),$$

and

$$(g(\omega, t, y_1, z, k) - g(\omega, t, y_2, z, k))(y_1 - y_2) \leq C|y_1 - y_2|^2.$$

The positive real C is called the λ-*constant* associated with driver g.

Nonlinear pricing and hedging of Game options

Definition 1: For each initial wealth x, a **super-hedge** against the game option is a pair (σ, φ) of a s.t. $\sigma \in \mathcal{T}$ and a strategy φ such that

$$V_t^{x,\varphi} \geq \xi_t,\ 0 \leq t \leq \sigma \text{ and } \quad V_\sigma^{x,\varphi} \geq \zeta_\sigma \text{ a.s.}$$

(Kifer 2000)

$\mathcal{A}(x) :=$ *set of all super-hedges associated with* x.

Definition 2: Define

$$u_0 := \inf\{x \in \mathbb{R},\ \exists(\sigma, \varphi) \in \mathcal{A}(x)\}.$$

- if inf is **attained** $\mapsto u_0$ is a **super-hedging price**.
- if inf is **not attained** $\mapsto u_0$ is a **"nearly" super-hedging price**.

Nonlinear pricing and hedging of Game options

Definition 3: A *natural price* for the seller of the game option is the ***g*-value** defined by

$$Y(0) := \inf_{\sigma \in \mathcal{T}} \sup_{\tau \in \mathcal{T}} \mathcal{E}^g[I(\tau, \sigma)],$$

where $I(\tau, \sigma) := \xi_\tau \mathbf{1}_{\tau \leq \sigma} + \zeta \mathbf{1}_{\sigma < \tau}$.

Nonlinear pricing and hedging of Game options

Aim

- Characterization of the superhedging price
- Characterization of the superhedging strategy

Nonlinear pricing and hedging of Game options

Main mathematical tool

Let ξ and ζ such that $\xi_t \leq \zeta_t$, $\xi_T = \zeta_T$ a.s. and satisfying the Mokobodzki's condition.

Definition (DRBSDE(g, ξ, ζ))

$$
\begin{aligned}
& -dY_t = g(t, Y_t, Z_t, k_t)dt + dA_t - dA'_t - Z_t dW_t - K_t dM_t \\
& Y_T = \xi_T, \\
& \xi_t \leq Y_t \leq \zeta_t, \ 0 \leq t \leq T \text{ a.s.},
\end{aligned}
$$

A and A' are nondecreasing RCLL predictable processes with $A_0 = 0$, $A'_0 = 0$ and such that

$$
\begin{cases}
\int_0^T (Y_{t-} - \xi_{t-}) dA_t = 0 \text{ a.s. and } \int_0^T (\zeta_{t-} - Y_{t-}) dA'_t = 0 \text{ a.s.} \\
dA_t \perp dA'_t.
\end{cases}
$$

Nonlinear pricing and hedging of Game options

Case I: ζ is left lower-s.c. along stopping times

Theorem (Characterization)

- *The superhedging price u_0 = g-value of the game option, i.e.*

$$u_0 = \inf_{\sigma \in \mathcal{T}} \sup_{\tau \in \mathcal{T}} \mathcal{E}^g_{0,\tau\wedge\sigma}[I(\tau,\sigma)] = \sup_{\tau \in \mathcal{T}} \inf_{\sigma \in \mathcal{T}} \mathcal{E}^g_{0,\tau\wedge\sigma}[I(\tau,\sigma)]$$

- *Let (Y, Z, K, A, A') be the solution of the DRBSDE(g, ξ, ζ). We have $u_0 = Y_0$. Let $\sigma^* := \inf\{t \geq 0,\ \ Y_t = \zeta_t\}$ and $\varphi^* := \Phi(Z, K)$. Then, (σ^*, φ^*) is a superhedge.*

Nonlinear pricing and hedging of Game options

Main step in the proof **:** ***Links between DRBSDEs and Generalized Dynkin Game (Dum.-Quenez-Sulem, EJP(2016)).***

If Y denotes the solution of the DRBSDE(g, ξ, ζ), we have:

$$Y_0 = \underbrace{\overbrace{\inf_\sigma \sup_\tau \mathcal{E}^g_{0,\tau\wedge\sigma}[I(\tau,\sigma)]}^{g\text{-value}} = \sup_\tau \inf_\sigma \mathcal{E}^g_{0,\tau\wedge\sigma}[I(\tau,\sigma)]}_{\text{Value Generalized Dynkin Game}}.$$

In other words, the solution of the doubly reflected BSDE corresponds to the *value function of an optimal stopping game with nonlinear expectation* (*Generalized Dynkin Game*).

Remark : There **does not a priori exist** τ^* such that (τ^*, σ^*) is a saddle point for the game problem.

Nonlinear pricing and hedging of Game options

Case II: ξ and ζ are only RCLL processes

When ζ is only RCLL, there does **not** necessarily exist a *super-hedge* against the option.

Theorem

The "nearly" superhedging price u_0 = g-value of the game option, i.e.

$$u_0 = \inf_{\sigma \in \mathcal{T}} \sup_{\tau \in \mathcal{T}} \mathcal{E}^g_{0,\tau\wedge\sigma}[I(\tau,\sigma)] = \sup_{\tau \in \mathcal{T}} \inf_{\sigma \in \mathcal{T}} \mathcal{E}^g_{0,\tau\wedge\sigma}[I(\tau,\sigma)]$$

For each $\varepsilon > 0$, let $\sigma_\varepsilon := \inf\{t \geq 0 : \ Y_t \geq \zeta_t - \varepsilon\}$. Let us consider the risky assets strategy $\varphi^* := \Phi(Z, K)$. We have

$$V_t^{Y_0,\varphi^*} \geq \xi_t, \ 0 \leq t \leq \sigma_\varepsilon \ \text{ a.s.} \quad \text{and} \quad V_{\sigma_\varepsilon}^{Y_0,\varphi^*} \geq \zeta_{\sigma_\varepsilon} - \varepsilon \ \text{ a.s.}$$

In other terms, the pair $(\sigma_\varepsilon, \varphi^*)$ is an ε-super-hedge for the initial capital amount Y_0.

Nonlinear pricing and hedging of Game options with uncertainty on the model

- Let $G : [0, T] \times \Omega \times \mathbb{R}^3 \times U \to \mathbb{R}$; $(t, \omega, z, k, u) \mapsto G(t, \omega, y, z, k, u)$, be a given measurable function (satisfying "good" assumptions).
- For each $u \in \mathcal{U}$, the associated driver is given by $g^u(t, \omega, y, z, k) := G(t, \omega, y, z, k, u_t(\omega))$.
- To each ambiguity parameter u, corresponds a market model $\mathcal{M}_u$ where the wealth process $V^{u,x,\varphi}$ satisfies

$$-dV_t^{u,x,\varphi} = G(t, V_t^{u,x,\varphi}, \varphi_t\sigma_t, -\varphi_t^2, u_t)dt - \varphi_t\sigma_t dW_t - \varphi_t^2 dM_t;$$
$$V_0^{u,x,\varphi} = x.$$

Nonlinear pricing and hedging of Game options with uncertainty on the model

- In the market model $\mathcal{M}_u$, the nonlinear pricing system is given by $\mathcal{E}^{g^u} := \{\mathcal{E}^{g^u}_{t,S},\ S \in [0,T], t \in [0,S]\}$.
- For each $u \in \mathcal{U}$, we denote by $Y^u(0)$ the g-value of the game option in the market model $\mathcal{M}_u$. It is equal to Y^u_0, where $(Y^u, Z^u, K^u, A^u, A'^u)$ is the unique solution of the $DRBSDE(g^u, \xi, \zeta)$.

Nonlinear pricing and hedging of Game options with uncertainty on the model

The seller being adverse to ambiguity, a *natural value* price of the game option, called g-value, is

$$Y(0) := \inf_{\sigma \in \mathcal{T}} \sup_{u \in \mathcal{U}} \sup_{\tau \in \mathcal{T}} \mathcal{E}^{g^u}_{0,\tau \wedge \sigma}[I(\tau, \sigma)].$$

Nonlinear pricing and hedging of Game options with uncertainty on the model

Definition 1: For each initial wealth x, a **super-hedge** against the game option is a pair (σ, φ) of a s.t. $\sigma \in \mathcal{T}$ and a portfolio strategy φ such that **for each** $u \in \mathcal{U}$, $V_t^{u,x,\varphi} \geq \xi_t$, $0 \leq t \leq \sigma$ and $V_\sigma^{u,x,\varphi} \geq \zeta_\sigma$ a.s.

$\mathcal{A}(x) :=$ *set of all super-hedges associated with* x.

Definition 2: Define

$$u_0 := \inf\{x \in \mathbb{R},\ \exists(\sigma, \varphi) \in \mathcal{A}(x)\}.$$

- if inf is **attained** $\mapsto$ u_0 is a **super-hedging price**.
- if inf is **not attained** $\mapsto$ u_0 is a **"nearly" super-hedging price**.

Nonlinear pricing and hedging of Game options with uncertainty on the model

Particular case: ζ is lower s.c. along stopping times

Theorem (Characterization)

The superhedging price u_0 *of the game option coincides with the* g-value *of the game option, that is*

$$u_0 = \inf_{\sigma \in \mathcal{T}} \sup_{u \in \mathcal{U}} \sup_{\tau \in \mathcal{T}} \mathcal{E}^{g^u}_{0,\tau\wedge\sigma}[I(\tau,\sigma)].$$

Let (Y, Z, K, A, A') *be the solution of the DRBSDE(g,ξ,ζ), where*

$$g(t,\omega,y,z,k) := \sup_{u \in U} g^u(t,\omega,y,z,k).$$

We have $u_0 = Y_0$.
Let $\sigma^* := \inf\{t \geq 0, \ \ Y_t = \zeta_t\}$ *and* $\varphi^* := \Phi(Z,K)$. *The pair* (σ^*, φ^*) *is a super-hedge.*

Nonlinear pricing and hedging of Game options with uncertainty on the model

Proof: A ***key point*** *is to identify the g-value to the solution Y of the DRBSDE(g, ξ, ζ).*

1. Optimization principle with BSDEs

- $\sup_u \sup_\tau \mathcal{E}^{g^u}_{0,\tau\wedge\sigma}[I(\tau,\sigma)] = \sup_\tau \sup_u \mathcal{E}^{g^u}_{0,\tau\wedge\sigma}[I(\tau,\sigma)] = \sup_\tau \mathcal{E}^{g}_{0,\tau\wedge\sigma}[I(\tau,\sigma)]$.

We get

$$\inf_\sigma \sup_u \sup_\tau \mathcal{E}^{g^u}_{0,\tau\wedge\sigma}[I(\tau,\sigma)] = \inf_\sigma \sup_\tau \mathcal{E}^{g}_{0,\tau\wedge\sigma}[I(\tau,\sigma)].$$

2. Links between DRBSDEs and Generalized Dynkin Game (Dum.-Quenez-Sulem, EJP(2016)).

$$\inf_\sigma \sup_\tau \mathcal{E}^{g}_{0,\tau\wedge\sigma}[I(\tau,\sigma)] = Y_0.$$

Nonlinear pricing and hedging of Game options with uncertainty on the model

Theorem (Interchange inf – sup)

We have the following equalities:

$$\inf_{\sigma \in \mathcal{T}} \sup_{u \in \mathcal{U}} \sup_{\tau \in \mathcal{T}} \mathcal{E}^{g^u}_{0,\tau\wedge\sigma}[I(\tau,\sigma)] = \sup_{u \in \mathcal{U}} \inf_{\sigma \in \mathcal{T}} \sup_{\tau \in \mathcal{T}} \mathcal{E}^{g^u}_{0,\tau\wedge\sigma}[I(\tau,\sigma)]$$

$$\sup_{u \in \mathcal{U}} \sup_{\tau \in \mathcal{T}} \inf_{\sigma \in \mathcal{T}} \mathcal{E}^{g^u}_{0,\tau\wedge\sigma}[I(\tau,\sigma)].$$

Financial interpretation:

The superhedging price of the game option in the case with ambiguity coincides with the supremum over $u \in \mathcal{U}$ of the (superhedging) prices Y_0^u corresponding to the market models $\mathcal{M}_u$.
When U is compact, there exists an optimal $u^* \mapsto$ *worst case scenario*.

Nonlinear pricing and hedging of Game options with uncertainty on the model

Main idea of the proof:

- In order to show equality (1): we establish an optimization principle for DRBSDEs (using a measurable selection theorem).
- Equality (2) is obtained using the Generalized Dynkin Games.

Nonlinear pricing and hedging of Game options with uncertainty on the model

General case: ξ and ζ are only RCLL processes

When ζ is only RCLL, there does **not** necessarily exist a *super-hedge* against the option.

Theorem

The g-value of the game option coincides with the "nearly" superhedging price, that is $Y_0 = u_0$.
For each $\varepsilon > 0$, let $\sigma_\varepsilon := \inf\{t \geq 0 : \quad Y_t \geq \zeta_t - \varepsilon\}$. Let us consider the risky assets strategy $\varphi^* := \Phi(Z, K)$. The pair $(\sigma_\varepsilon, \varphi^*)$ is an ε-super-hedge *for the seller.*

Nonlinear pricing and hedging of Game options with uncertainty on the model

Example with ambiguity on the default probability. Suppose that G is defined by:

$$G(t,\omega,u,y,z,k) = \beta(t,\omega,u)z + \gamma(t,\omega,u)k + f(t,\omega,z,k),$$

with β, γ bounded. Let Q^u be the probability measure which admits Z_T^u as density with respect to P, where (Z_t^u) is the solution of the following SDE:

$$dZ_t^u = Z_t^u[\beta(t,u_t)dW_t + \gamma(t,u_t)dM_t]; \quad Z_0^u = 1.$$

Under Q^u, $W_t^u := W_t - \int_0^t \beta(s,u_s)ds$ is a Brownian motion and $M_t^u := M_t - \int_0^t \lambda_s(1+\gamma(s,u_s))ds$ is a martingale independant of W^u.

Nonlinear pricing and hedging of Game options with uncertainty on the model

For each $u \in \mathcal{U}$, the market model $\mathcal{M}_u$ can be seen as a market model associated with *a probability* Q^u, where the dynamics of the wealth process can be written

$$-dV_t = f(t, V_t, Z_t, K_t)dt - Z_t dW_t^u - K_t dM_t^u.$$

The g^u-evaluation of an option with maturity S and payoff $\xi \in L^p(\mathcal{F}_S)$ with $p > 2$, can be written

$$\mathcal{E}_{0,S}^u(\xi) = \mathcal{E}_{Q^u,0,S}^f(\xi).$$

The nonlinear price system in the market model $\mathcal{M}_u$ is the f-evaluation under the *probability measure* Q^u.

THANK YOU FOR YOUR ATTENTION!