Root’s solution to Skorokhod embedding

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The Skorokhod embedding problem

- $B$ standard, real-valued Brownian motion
- $\mu$ a probability measure on $(\mathbb{R}, B)$

Skorokhod embedding problem (SEP) : find stopping time $\tau$ such that

$$B_\tau \sim \mu \quad \text{and} \quad B^\tau = (B_{\tau \wedge t}) \text{ is uniformly integrable}$$

For $B_0 = 0$, there exists a solution as long as $\mu$ has a first moment and is centered. In fact, many different solutions/constructions, e.g.

- Skorokhod ’61, Dubins ’68, Root ’68, Rost ’71, Monroe ’72, Azema, Yor, Bertoin–Le Jan, and many others.
Outline

1. Root’s barrier for 1D Brownian motion

2. Root’s barrier for general Markov processes
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Root’s solution to the SEP

We call barrier a subset $R \subset [0, +\infty) \times \mathbb{R}$ s.t.

$$(t, x) \in R, \quad s \geq t \quad \Rightarrow \quad (s, x) \in R.$$ 

Note that $R$ is a closed barrier iff for some l.s.c. function $r : \mathbb{R} \to [0, \infty]$

$$R = \{(t, x) \mid t \geq r(x)\}$$

**Theorem (Root, 1968)**

Let $\mu$ be a centered probability measure with first moment. Then there exists a closed barrier $R$ such that

$$\tau = \inf \{t \geq 0 \mid (t, B_t) \in R\}$$

solves the Skorokhod embedding problem for $\mu$.

**Theorem (Rost, 1976)**

Root’s embedding minimizes $\mathbb{E}[F(\tau)]$ for all convex increasing $F$, among other solutions to the SEP.
Example 1: $\mu = \mathcal{N}(0, 1)$
Example 2: $\mu = \frac{1}{2} \delta_{-1} + \frac{1}{2} \delta_1$
How to compute the barrier?

Root & Rost say nothing on how to compute the barrier $R$. Recently, regain of interest due to connections with model-free finance, starting with Dupire (2005).

Solution expressed in terms of potential functions:

**Definition**

For probability measure $\mu$ on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ with finite first moment, let

$$u_\mu(x) := -\int_{\mathbb{R}} |x - y| \, \mu(dy)$$

We call $u_\mu$ the **potential function** of $\mu$.

- $u_{\delta_0}(x) = -|x|$ and $u_{\delta_0}(x) \geq u_\mu(x)$ for any centered probability measure $\mu$.
- More generally, there exists a solution to the SEP$(\nu, \mu)$

$$B_0 \sim \nu, \quad B_\tau \sim \mu, \quad (B_{t\wedge \tau})_{t \geq 0} \text{ u.i.}$$

if and only if $u_\nu \geq u_\mu$. 
The Root’s barrier as a free boundary

**Theorem (Dupire ’05, Cox&Wang ’12)**

Let $\nu, \mu$ be probability measures with $u_\nu \geq u_\mu$. Then the Root’s barrier $R = R(\nu, \mu)$ is the free boundary of the solution of the obstacle problem

$$\begin{aligned}
\min \left( u - u_\mu, \partial_t u - \frac{1}{2} \Delta u \right) &= 0, \\
 u(0, x) &= u_\nu(x)
\end{aligned}$$

i.e. $R = \{(t, x) : u(t, x) = u_\mu(x)\}$.

In addition, if $\tau$ is the Root stopping time corresponding to $(\nu, \mu)$, then

$$u(t, x) := -\mathbb{E}^\nu[|x - B_{t\wedge \tau}|].$$
Example

Figure: $\nu = \delta_0$ and $\mu = \frac{2}{7} \delta_{-1} + \frac{1}{4} \delta_{-\frac{1}{4}} + \frac{13}{28} \delta_{\frac{3}{4}}$. 

\[
\frac{1}{2} \begin{align*}
&-1 & -0.5 & 0.5 & \quad x \\
&1.0 & -1.0 & -0.5 & 0.5 \\
&x & -\nu & \mu & -\nu & \mu & u(t, x) \\
&-1.0 & -1.0 & -0.5 & 0.5 & x & \quad t
\end{align*}
\]
Litterature overview

- Root ('68) : probabilistic proof of existence.
- Röst ('76) : Different approach. First prove existence of an optimal embedding. Then deduce that it must be necessarily the hitting time of a barrier. (Recently generalized by Beiglboeck, Cox, Huesmann).
- Cox & Wang ('12) : martingale interpretation of the optimality of Root’s solution.
- G., Oberhauser, dos Reis ('14) : direct proof of existence and characterization based on maximum principle.
- Cox, Obloj, Touzi ('15) : multi-marginal case.

Variants and extensions :
- ”Reversed” barrier : Röst ('7x), Chacon ('85), McConnell ('91), Cox-Wang ('15), de Angelis ('16)...
- 1D diffusions $dX_t = \sigma(t, X_t)dB_t,$...
Application: bounded Brownian increments

Standard way to simulate a Brownian path: fix $\Delta t > 0$, and then on the grid $\{k\Delta t, \ k = 0, 1, \ldots\}$

$$W_{(k+1)\Delta t} = W_t + \sqrt{\Delta t}Y_k, \ Y_k \sim \mathcal{N}(0, 1),$$

and then take $\hat{W}$ piecewise-linear approximation. Then $\|\hat{W} - W\|$ is small in probability, but no almost sure bounds.

This may be problematic for some applications (e.g. exit distribution from a space-time domain, ...)

→ replace the deterministic time-increment by exit-times from a chosen (space-time) set

→ (Root's solution to ) Skorokhod embedding: choose target space distribution (e.g. $\mu = \mathcal{U}[-1, 1]$), then compute space-time barrier $R$ such that

$$B_{\tau R} \sim \mu$$
The Root’s barrier as solution to an integral equation

**Theorem (G&Mijatovic&Oberhauser 13)**

Assume that \( r \) is the lsc function corresponding to \( R(\delta_0, \mu) \), where \( \mu = \mathcal{U}[-1, 1] \). Then \( r \) is the unique continuous solution on \([-1, 1]\) to

\[
\begin{align*}
    u_\nu(x) - u_\mu(x) &= g(r(x), x) \\
    &- \int_x^1 (g(r(x) - r(y), x - y) + g(r(x) - r(y), x + y)) \, dy
\end{align*}
\]

where \( g(t, y) = \int_0^t p_s(y) \, ds \), \( p_s \) density of Brownian motion at time \( s \).
Example: Numerical resolution for $\mu = \mathcal{U}([-1, 1])$

Discretize $[0, 1]$ with $0 = x_0 \leq \ldots \leq x_i = \frac{i}{n} \leq \ldots \leq x_n = 1$. Then take $r_n = 0$, and inductively

$$u_{\mu}(ih) - u_{\delta}(ih) = g(r_i, ih) - \sum_{j=i+1}^{n} \frac{1}{2} \left( g(r_i - r_j, (i-j)h) + g(r_i - r_j, (i+j)h) \right).$$
Root's barrier for 1D Brownian motion

\[ \mu = U[-1,1], 100 \text{ Brownian trajectories} \]

\[ B_t \wedge \tau \]

\[ r(x) \text{ evaluated at 100 samples from } \mu = U[-1,1] \]
Application: Random Walk over Root’s barriers

(Müller ’56: Random walk on spheres for standard multi-dim BM, then many others...)

\[ D = \bigcup_{t \in (0, T)} \{t\} \times (a_t, b_t) \]

where \( T \in (0, \infty) \) is fixed, \( a, b \in C^1((0, T), \mathbb{R}) \) and \( a_t < b_t \) on \((0, T)\).

Objective: compute \( u \) solution to

\[
\begin{align*}
\partial_t u + \frac{1}{2} \Delta u &= 0 \text{ on } D \\
u(t, x) &= g(t, x) \text{ on } PD
\end{align*}
\]

Let \( r = r(\delta_0, \mathcal{U}[-1, 1]) \) (computed beforehand).

For all \((t, x)\) in \( D \), \( \rho(t, x) \) largest \( \rho \) s.t. \( B_{t,x}^\rho \subset D \), where

\[ B^\rho = \{(t + \rho^2 s, x + \rho y) \text{ s.t. } r(y) \leq s\} \cdot \]

\( \rho(t, x) \sim \) (parabolic) distance from \((t, x)\) to the boundary.
Random walk over Root’s barriers

Algorithm:
- \((T_0, X_0) = (t, x)\)
- Inductively, let
  \[
  \begin{align*}
  X_{k+1} &= X_k + \rho^2(T_k, X_k) U_k, \\
  T_{k+1} &= T_k + \rho(T_k, X_k) r(U_k),
  \end{align*}
  \]
  where \(U_k\) sequence of i.i.d. \(\mathcal{U}[-1, 1]\).
- Stop at \(K_\delta = \inf\{k \geq 0, \rho(T_k, X_k) \leq \delta\}\).

Proposition

\[\exists c_1, c_2 \text{ such that for every } \delta > 0\]
\[|\mathbb{E}_{t,x}[g(T_{K_\delta}, X_{K_\delta})] - u(t, x)| \leq c_1 \delta.\]

In addition, the average number of steps satisfies
\[\mathbb{E}_{t,x}[K_\delta] \leq c_2 (1 + \log(1/\delta)) \quad \forall (t, x) \in \mathcal{D}\]
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2 Root’s barrier for general Markov processes
Motivation: stochastic Taylor expansion

Let $X$ be the solution to the Stratonovich SDE

$$dX_t = \sum_{i=1}^{d} V_i(X_t) \circ dB^i_t$$

Then for smooth $f$, all stopping time $\tau \leq 1$,

$$f(X_t) = f(X_0) + \sum_i (V_i f)(X_0) (B^i_\tau - B^i_0) + \sum_{i,j} (V_i V_j f)(X_0) \left( \int_0^\tau B^i_s \circ dB^j_s \right)$$

$$+ \ldots + \sum_{i_1,\ldots,i_k} (V_{i_1} \ldots V_{i_k} f)(X_0) \left( \int_0 \leq s_1 \leq \ldots \leq s_k \leq \tau \circ dB^{i_1}_{s_1} \ldots \circ dB^{i_k}_{s_k} \right) + R_{0,\tau,k}$$

where $R_{0,\tau,k} = O(\tau^{k+1/2})$. $\rightarrow$ higher order discretization schemes

$\rightarrow$ Consider embedding for Enhanced Brownian Motion:

$$B_t = \left( 1, B_t, \int_0^t B_s \otimes \circ dB_s, \ldots, \int_0 \leq t_1 \leq \ldots \leq t_n \leq t dB_{t_1} \otimes \cdots \otimes dB_{t_n} \right)$$
Skorokhod embedding for general Markov processes

Let $X$ be a transient Markov process on $(E, \mathcal{E})$ with semigroup $(P_t)_{t \geq 0}$.

Skorokhod Embedding: given two probability measures $\mu, \nu$ on $E$,

Find a stopping time $\tau$ s.t. $X_\tau \sim \mu$, $\mathbb{P}^\nu$-a.s. SEP($\nu, \mu$)

**Question:**

- When is there a solution $\tau$ to SEP($\nu, \mu$) given by the hitting time of a barrier $\tau = \inf \{t \geq 0, \ (t, X_t) \in R\}$?
- When such a stopping time exists, how do we determine $R$?
(Abstract) classical existence results

Potential kernel $U = \int_0^\infty P_t dt$, i.e. $\mu U(A) = \mathbb{E}^\mu \left[ \int_0^\infty 1_A(X_t) dt \right]$. 

**Theorem (Rost '71)**

There exists a solution to $SEP(\nu, \mu)$ possibly requiring external randomization if and only if $\nu U \geq \mu U$.

We are looking to embed $\mu$ as hitting time of a barrier: no additional randomization allowed! When is there a solution to $SEP(\nu, \mu)$ given by a natural stopping time? Counter-examples:

- $X_t = X_0 + t$, $\nu = \delta_0$. Then for any $\mu$ supported on $\mathbb{R}_+$, there exists a solution, but not for natural s.t. unless $\mu = \delta_x$.
- $X$ multi-dim. BM, $\nu = \delta_0$, $\mu = \frac{1}{2} \delta_0 + \frac{1}{2} U(S(1))$. Any solution to $(SEP(\nu, \mu))$ requires external randomization, by 0-1 law and the fact that points are polar for multi-dim BM.

(Existence results for Skorokhod embedding in the class of natural stopping times due to Falkner ('82, '83) Falkner & Fitzsimmons ('91).)
Root’s barrier for Markov processes

Assumptions:

1. $X$ is a standard process (in the sense of Blumenthal Getoor) in duality with a standard process $\hat{X}$ with respect to a measure $\xi$. In particular, this implies that $\frac{\nu U}{d\xi} =: \nu \hat{U}$ exists.

In addition, assume that that

$$P_t(x, dy) = p_t(x, y)\xi(dy), \quad \hat{P}_t(dx, y) = p_t(x, y)\xi(dx),$$

and $t \mapsto p_t(x, y)$ is continuous in $t$ (with some uniformity condition).

2. Any semipolar set is polar.
   (Let $J_A : \{ t > 0, X_t \in A \}$, then $A$ is polar if $J_A = \emptyset$ a.s., and semipolar if $J_A$ is countable a.s.)
Root’s barrier for Markov processes

Theorem (Bayer, G., Oberhauser (in progress))

Let \( \nu, \mu \) be probability measures with \( \nu U \supseteq \mu U \), and such that \( \mu \) charges no semipolar set. Then:

- Under Assumption (1), there exists a barrier \( R \) such that
  \[
  T_R := \inf\{t > 0, (t, X_t) \in R\}
  \]
  embeds \( \mu \) into \( \nu \).

- Under Assumptions (1) and (2), one can take \( R \) given as the free boundary to the obstacle problem with obstacle
  \[
  \nu \hat{U} 1_{\{t \leq 0\}} + \mu \hat{U} 1_{\{t > 0\}}.
  \]

Proof of existence: strongly relies on results of Rost and R.M. Chacon.

One can write a more general condition on \( \mu \) such that there exists a barrier embedding.
Enhanced brownian motion

Let $B$ be standard BM on $\mathbb{R}^d$. We consider the Markov process

$$B_t = \left(1, B_t, \int_0^t B_s \otimes \circ dB_s, \ldots, \int_0 \leq t_1 \leq \ldots \leq t_n \leq t dB_{t_1} \otimes \cdots \otimes \circ dB_{t_n}\right)$$

Brownian motion on nilpotent Lie group $G_{n,d} = \exp g_{n,d}$ (definition of elements of $G_{n,d}$ reflect chain rule, e.g. $\int B^1 \circ dB^1 = \frac{1}{2}(B^1)^2$).

To simplify, fix now $n = d = 2$.

Log-coordinates

$$(X_t, Y_t, A_t) = \left(B^1_t, B^2_t, \frac{1}{2} \int_0^t B^1_s dB^2_s - B^2_s dB^1_s\right).$$

Group law

$$(x, y, a) \ast (x', y', a') = \left(x + x', y + y', a + a' + \frac{1}{2} (xy' - x'y)\right).$$

$B$ has generator $\frac{1}{2} \Delta_G = \frac{1}{2} (X^2_1 + X^2_2)$ where in coordinates

$$X_1 = \partial_x - \frac{y}{2} \partial_a, \quad X_2 = \partial_y + \frac{x}{2} \partial_a.$$ 

The sub-laplacian $\Delta_G$ is hypoelliptic on $G_{n,d}$. 
Potential kernel:

$$\mu \hat{U}(h) := \int_{G_{d,n}} \mu(dg)u(g, h),$$

where $u$ is the fundamental solution $\Delta_G u(\cdot, h) = -\delta_h$, and is given by

$$u(g, h) = N(g \ast h^{-1})^{-Q+2},$$

where $Q = Q(d, n)$ and $N$ is an “homogeneous norm” on $G_{d,n}$. Special cases:

- $n = 1, d \geq 3$: $Q = d$, $N$ Euclidean norm on $\mathbb{R}^d$.
- $n = d = 2$:

  $$Q = 4, \quad N(x, y, a) = c_N \left( (x^2 + y^2)^2 + 16a^2 \right)^\frac{1}{4}. $$

Enhanced Brownian motion $B$ satisfies the assumptions needed for existence and characterization of the Root barrier.
We need to find measures \( \mu \) such that \( \mu \hat{U} \leq \delta_0 \hat{U} \) (if possible with a density which is easy to sample from):

- Very explicit formulae (Gaveau '77, Bonfiglioli-Lanconelli '03) for exit measures for \( B \) from “sphere” \( S_N(r) = \{ g \in G_{d,n}, \quad N(g) = r \} \)

\( \rightarrow \) one can then take linear combinations of those measures (\( \sim \) radial measures).

**Proposition**

Let \( \tilde{\mu} \) be any probability measure on \( (0, \infty) \). Let

\[
\mu = \int_0^\infty \tilde{\mu}(dr) \frac{c}{r^{Q-1}} \int_{S_N(r)} \frac{|\nabla H N|^2(g)}{|\nabla N|(g)} d\sigma(g),
\]

where \( \sigma \) is the surface measure, and \( |\nabla H N|^2 := \sum_{i=1}^n |X_i(N)|^2 \).

Then one has

\[
\mu \hat{U} \leq \delta \hat{U}.
\]
For instance, for $n = d = 2$, one can take

$$d\mu(x, y, a) = \frac{4}{\pi} 1_{(x^2+y^2)^2+16a^2<1} \frac{x^2 + y^2}{(x^2 + y^2)^2 + 16a^2}^{1/2} \, dx \, dy \, da.$$ 

Then $\mu \hat{U} \leq \delta_0 \hat{U}$, and $\mu \hat{U}$ is bounded and continuous.

→ one can then numerically compute $\mu \hat{U}$, and solve the obstacle problem

$$\left\{ \begin{array}{l}
\min\{ (\partial_t - \frac{1}{2} \Delta G) f, f - \mu \hat{U} \} = 0 \quad \text{on (0, } \infty) \times G, \\
 f(0, \cdot) = \delta_0 \hat{U}
\end{array} \right.$$ 

to obtain the Root’s barrier.

Note that due to symmetry, $x, y$ will only appear through $x^2 + y^2$. So we only need to solve a PDE in 2 space dimensions (and then store a 2-dim array for barrier function $r$).

Work in progress: numerical implementation, application to simulation of SDEs with bounded increments (in time, space, area),...
Conclusion

Root’s solution to the Skorokhod embedding:

- conceptually simple (just a hitting time)
- can be computed efficiently (obstacle problem, integral equation)
- many applications (model-free finance, numerics,...)