Root's solution to Skorokhod embedding

Paul Gassiat

CEREMADE, Université Paris Dauphine

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Joint works with C. Bayer (WIAS Berlin), A. Mijatovic (Imperial College), H. Oberhauser (Oxford), and G. dos Reis (Edinburgh)

The Skorokhod embedding problem

- B standard, real-valued Brownian motion
- μ a probability measure on $(\mathbb{R}, \mathcal{B})$

Skorokhod embedding problem (SEP) : find stopping time τ such that

$$B_\tau \sim \mu \qquad \text{and } B^\tau = (B_{\tau\wedge t}) \text{ is uniformly integrable}$$

For $B_0 = 0$, there exists a solution as long as μ has a first moment and is centered. In fact, *many* different solutions/constructions, e.g.

- Skorokhod '61, Dubins '68, Root '68, Rost '71, Monroe '72, Azema, Yor, Bertoin–Le Jan, and *many* others.

Outline

1. Root's barrier for $1D$ Brownian motion
2. Root's barrier for general Markov processes

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Root's solution to the SEP

We call *barrier* a subset $R \subset [0, +\infty) \times \mathbb{R}$ s.t.

$$(t, x) \in R, \quad s \geqslant t \qquad \Rightarrow \qquad (s, x) \in R.$$

Note that R is a closed barrier iff for some l.s.c. function $r : \mathbb{R} \to [0, \infty]$

$$R = \{(t, x) \mid t \geqslant r(x)\}$$

Theorem (Root, 1968)

Let μ be a centered probability measure with first moment. Then there exists a closed barrier R such that

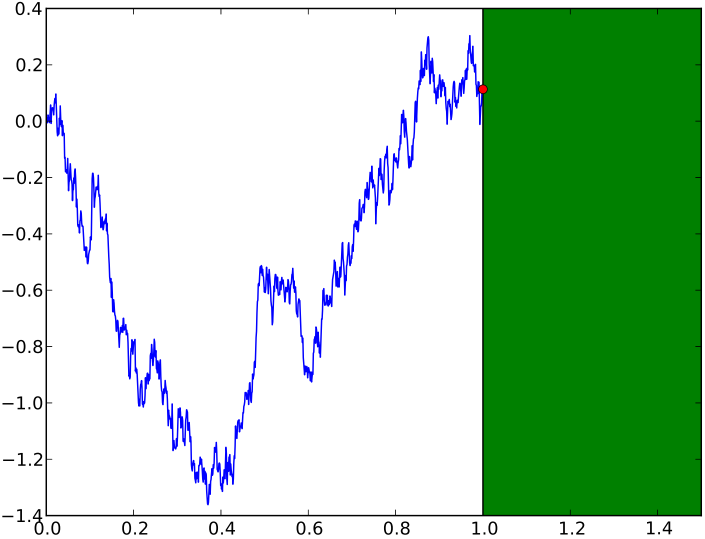
$$\tau \;=\; \inf \{t \geqslant 0 \mid (t, B_t) \in R\}$$

solves the Skorokhod embedding problem for μ.

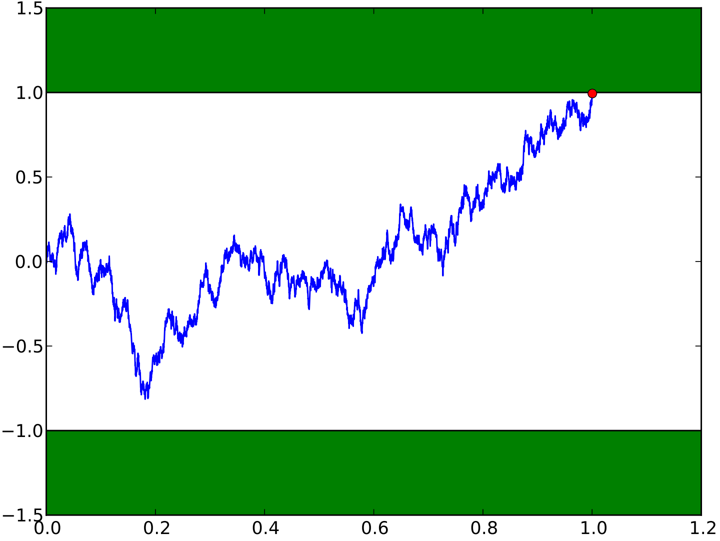
Theorem (Rost, 1976)

Root's embedding minimizes $\mathbb{E}[F(\tau)]$ for all convex increasing F, among other solutions to the SEP.

Example 1 : $\mu = \mathcal{N}(0, 1)$



Example 2 : $\mu = \frac{1}{2}\delta_{-1} + \frac{1}{2}\delta_1$



How to compute the barrier ?

Root & Rost say nothing on how to compute the barrier R.
Recently, regain of interest due to connections with model-free finance, starting with Dupire (2005).
Solution expressed in terms of potential functions :

Definition

For probability measure μ on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ with finite first moment, let

$$u_\mu(x) := -\int_{\mathbb{R}} |x-y|\,\mu(dy)$$

We call u_μ the **potential function** of μ.

- $u_{\delta_0}(x) = -|x|$ and $u_{\delta_0}(x) \geqslant u_\mu(x)$ for any centered probability measure μ.
- More generally, there exists a solution to the SEP(ν, μ)

$$B_0 \sim \nu, \quad B_\tau \sim \mu, \quad (B_{t\wedge\tau})_{t\geqslant 0} \text{ u.i.}$$

if and only if $u_\nu \geqslant u_\mu$.

The Root's barrier as a free boundary

Theorem (Dupire '05, Cox&Wang '12)

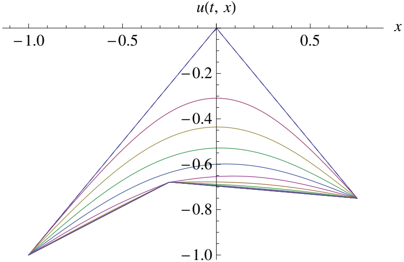
Let ν, μ be probability measures with $u_\nu \geqslant u_\mu$. Then the Root's barrier $R = R(\nu, \mu)$ is the free boundary of the solution of the obstacle problem

$$\begin{cases} \min\left(u - u_\mu, \partial_t u - \frac{1}{2}\Delta u\right) & = & 0, \\ u\left(0, x\right) & = & u_\nu\left(x\right) \end{cases}$$

i.e. $R = \{(t, x) : u\left(t, x\right) = u_\mu\left(x\right)\}$.
In addition, if τ is the Root stopping time corresponding to (ν, μ), then

$$u(t, x) := -\mathbb{E}^\nu\left[|x - B_{t\wedge\tau}|\right].$$

Example



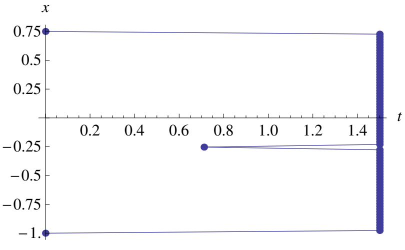


Figure: $\nu = \delta_0$ and $\mu = \frac{2}{7}\delta_{-1} + \frac{1}{4}\delta_{-\frac{1}{4}} + \frac{13}{28}\delta_{\frac{3}{4}}$.

Litterature overview

- Root ('68) : probabilistic proof of existence.
- Röst ('76) : Different approach. First prove existence of an optimal embedding. Then deduce that it must be necessarily the hitting time of a barrier.
  (Recently generalized by Beiglboeck, Cox, Huesmann).
- Cox & Wang ('12) : martingale interpretation of the optimality of Root's solution.
- G., Oberhauser, dos Reis ('14) : direct proof of existence and characterization based on maximum principle.
- Cox, Obloj, Touzi ('15) : multi-marginal case.

Variants and extensions :

- "Reversed" barrier : Röst ('7x), Chacon ('85), McConnell ('91), Cox-Wang ('15), de Angelis ('16)...
- $1D$ diffusions $dX_t = \sigma(t, X_t)dB_t$,...

Application : bounded Brownian increments

Standard way to simulate a Brownian path : fix $\Delta t > 0$, and then on the grid $\{k\Delta t,\ k = 0, 1, \dots\}$

$$W_{(k+1)\Delta_t} = W_t + \sqrt{\Delta t}\, Y_k, \quad Y_k \sim \mathcal{N}(0,1),$$

and then take $\hat{W}$ piecewise-linear approximation. Then $\|\hat{W} - W\|$ is small in probability, but no almost sure bounds.
This may be problematic for some applications (e.g. exit distribution from a space-time domain,...)
$\longrightarrow$ replace the deterministic time-increment by exit-times from a chosen (space-time) set
$\longrightarrow$ (Root's solution to) Skorokhod embedding : choose target space distribution (e.g. $\mu = \mathcal{U}[-1,1]$), then compute space-time barrier R such that

$$B_{\tau^R} \sim \mu$$

The Root's barrier as solution to an integral equation

Theorem (G&Mijatovic&Oberhauser 13)

Assume that r is the lsc function corresponding to $R(\delta_0, \mu)$, where $\mu = \mathcal{U}[-1,1]$. Then r is the unique continuous solution on $[-1,1]$ to

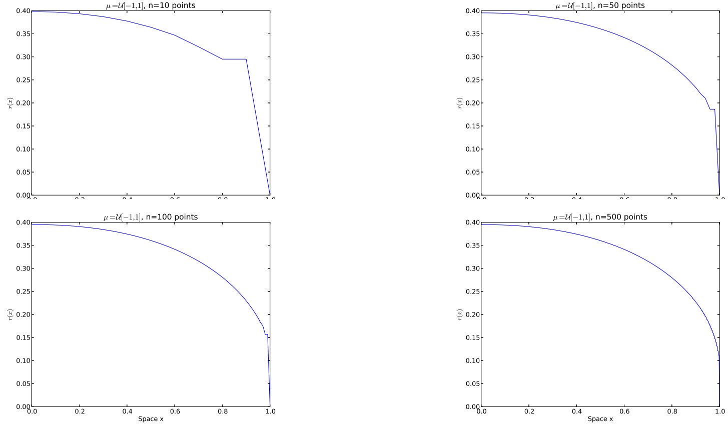
$$u_\nu(x) - u_\mu(x) = g(r(x), x) - \int_x^1 \left(g(r(x)-r(y), x-y) + g(r(x)-r(y), x+y)\right) \mathrm{d}y$$

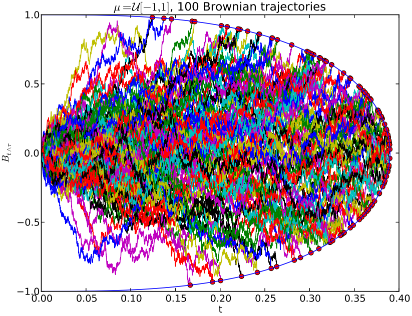
where $g(t,y) = \int_0^t p_s(y) ds$, p_s density of Brownian motion at time s.

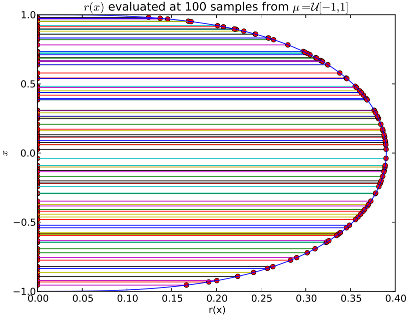
Example : Numerical resolution for $\mu = \mathcal{U}([-1,1])$

Discretize $[0,1]$ with $0 = x_0 \leqslant \ldots \leqslant x_i = \frac{i}{n} \leqslant \ldots \leqslant x_n = 1$. Then take $r_n = 0$, and inductively

$$u_\mu(ih) - u_\delta(ih) = g(r_i, ih) - \sum_{j=i+1}^{n} \frac{1}{2}\left(g\left(r_i - r_j, (i-j)h\right) + g\left(r_i - r_j, (i+j)h\right)\right).$$




$\mu = \mathcal{U}[-1,1]$, 100 Brownian trajectories
$B_{t\wedge\tau}$
t
1.0
0.5
0.0
−0.5
−1.0
0.00
0.05
0.10
0.15
0.20
0.25
0.30
0.35
0.40


$r(x)$ evaluated at 100 samples from $\mu = \mathcal{U}[-1,1]$
x
r(x)
1.0
0.5
0.0
−0.5
−1.0
0.00
0.05
0.10
0.15
0.20
0.25
0.30
0.35
0.40

Application : Random Walk over Root's barriers

(Müller '56 : Random walk on spheres for standard multi-dim BM, then many others...)

$$\mathcal{D} = \bigcup_{t \in (0,T)} \{t\} \times (a_t, b_t)$$

where $T \in (0, \infty)$ is fixed, $a, b \in C^1((0, T), \mathbb{R})$ and $a_t < b_t$ on $(0, T)$.
Objective : compute u solution to

$$\begin{cases} \partial_t u + \frac{1}{2}\Delta u & = & 0 \text{ on } \mathcal{D} \\ u(t,x) & = & g(t,x) \text{ on } \mathcal{PD} \end{cases}$$

Let $r = r(\delta_0, \mathcal{U}[-1,1])$ (computed beforehand).
For all (t,x) in D, $\rho(t,x)$ largest ρ s.t. $B^{\rho}_{t,x} \subset \mathcal{D}$, where

$$B^{\rho} = \left\{(t + \rho^2 s, x + \rho y) \quad \text{s.t. } r(y) \leqslant s\right\}.$$

$\rho(t,x) \sim$ (parabolic) distance from (t,x) to the boundary.

Random walk over Root's barriers

Algorithm :

- $(T_0, X_0) = (t, x)$
- Inductively, let

$$\begin{cases} X_{k+1} = X_k + \rho^2(T_k, X_k)U_k, \\ T_{k+1} = T_k + \rho(T_k, X_k)r(U_k), \end{cases}$$

  where U_k sequence of i.i.d. $\mathcal{U}[-1, 1]$.
- Stop at $K_\delta = \inf\{k \geqslant 0, \quad \rho(T_k, X_k) \leqslant \delta\}$.

Proposition

$\exists c_1, c_2$ such that for every $\delta > 0$

$$|\mathbb{E}_{tx}[g(T_{K_\delta}, X_{K_\delta})] - u(t, x)| \leqslant c_1\delta.$$

In addition, the average number of steps satisfies

$$\mathbb{E}_{t,x}[K_\delta] \leqslant c_2(1 + \log(1/\delta)) \quad \forall (t, x) \in \mathcal{D}$$

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2. Root's barrier for general Markov processes

Motivation : stochastic Taylor expansion

Let X be the solution to the Stratonovich SDE

$$dX_t = \sum_{i=1}^{d} V_i(X_t) \circ dB_t^i$$

Then for smooth f, all stopping time $\tau \leqslant 1$,

$$f(X_t) = f(X_0) + \sum_i (V_i f)(X_0)\left(B_\tau^i - B_0^i\right) + \sum_{i,j}(V_i V_j f)(X_0)\left(\int_0^\tau B_s^i \circ dB_s^j\right)$$
$$+ \ldots + \sum_{i_1,\ldots,i_k} \left(V_{i_1}\ldots V_{i_k} f\right)(X_0)\left(\int_{0 \leqslant s_1 \leqslant \ldots \leqslant s_k \leqslant \tau} \circ dB_{s_1}^{i_1} \cdots \circ dB_{s_k}^{i_k}\right) + R_{0,\tau,k}$$

where $R_{0,\tau,k} = O(\tau^{\frac{k+1}{2}})$. $\longrightarrow$ higher order discretization schemes
$\longrightarrow$ Consider embedding for *Enhanced Brownian Motion* :

$$\mathbf{B}_t = \left(1, B_t, \int_0^t B_s \otimes \circ dB_s, \ldots, \int_{0 \leqslant t_1 \leqslant \ldots \leqslant t_n \leqslant t} dB_{t_1} \otimes \cdots \otimes \circ dB_{t_n}\right)$$

Skorokhod embedding for general Markov processes

Let X be a transient Markov process on $(E, \mathcal{E})$ with semigroup $(P_t)_{t \geqslant 0}$
Skorokhod Embedding : given two probability measures μ, ν on E,

$$\text{Find a stopping time } \tau \text{ s.t. } X_\tau \sim \mu, \quad \mathbb{P}^\nu - \text{a.s.} \qquad \text{SEP}(\nu, \mu)$$

Question :

- When is there a solution τ to $SEP(\nu, \mu)$ given by the hitting time of a barrier $\tau = \inf\{t \geqslant 0, \ | \ (t, X_t) \in R\}$?
- When such a stopping time exists, how do we determine R ?

(Abstract) classical existence results

Potential kernel $U = \int_0^\infty P_t dt$, i.e. $\mu U(A) = \mathbb{E}^\mu \left[\int_0^\infty 1_A(X_t) dt \right]$.

Theorem (Rost '71)

There exists a solution to SEP(ν, μ) **possibly requiring external randomization** *if and only if $\nu U \geqslant \mu U$.*

We are looking to embed μ as hitting time of a barrier : no additional randomization allowed ! When is there a solution to SEP(ν, μ) given by a **natural** stopping time ?

Counter-examples :

- $X_t = X_0 + t$, $\nu = \delta_0$. Then for any μ supported on $\mathbb{R}_+$, there exists a solution, but not for natural s.t. unless $\mu = \delta_x$.
- X multi-dim. BM, $\nu = \delta_0$, $\mu = \frac{1}{2}\delta_0 + \frac{1}{2}\mathcal{U}(S(1))$. Any solution to (SEP$(\nu, \mu)$) requires external randomization, by 0-1 law and the fact that points are polar for multi-dim BM.

(Existence results for Skorokhod embedding in the class of **natural** stopping times due to Falkner ('82, '83) Falkner & Fitzsimmons ('91).)

Root's barrier for Markov processes

Assumptions :

1. X is a standard process (in the sense of Blumenthal Getoor) in duality with a standard process $\hat{X}$ with respect to a measure ξ. In particular, this implies that $\frac{\nu U}{d\xi} =: \nu \hat{U}$ exists.

   In addition, assume that that

   $$P_t(x, dy) = p_t(x, y)\xi(dy), \quad \hat{P}_t(dx, y) = p_t(x, y)\xi(dx),$$

   and $t \mapsto p_t(x, y)$ is continuous in t (with some uniformity condition).

2. Any semipolar set is polar.
   (Let $J_A : \{t > 0, X_t \in A\}$, then A is *polar* if $J_A = \emptyset$ a.s., and *semipolar* if J_A is countable a.s.)

Root's barrier for Markov processes

Theorem (Bayer, G., Oberhauser (in progress))

Let ν, μ be probability measures with $\nu U \geqslant \mu U$, and such that μ charges no semipolar set. Then :

- *Under Assumption (1), there exists a barrier R such that*

$$T_R := \inf\{t > 0,\ (t, X_t) \in R\}$$

  embeds μ into ν.
- *Under Assumptions (1) and (2), one can take R given as the free boundary to the obstacle problem with obstacle $\nu \hat{U} 1_{\{t \leqslant 0\}} + \mu \hat{U} 1_{\{t > 0\}}$.*

- Proof of existence : strongly relies on results of Rost and R.M. Chacon.
- One can write a more general condition on μ such that there exists a barrier embedding.

Enhanced brownian motion

Let B be standard BM on $\mathbb{R}^d$. We consider the Markov process

$$\mathbf{B}_t = \left(1, B_t, \int_0^t B_s \otimes \circ dB_s, \dots, \int_{0 \leqslant t_1 \leqslant \dots \leqslant t_n \leqslant t} dB_{t_1} \otimes \cdots \otimes \circ dB_{t_n}\right)$$

Brownian motion on nilpotent Lie group $G_{n,d} = \exp \mathfrak{g}_{n,d}$ (definition of elements of $G_{n,d}$ reflect chain rule, e.g. $\int B^1 \circ dB^1 = \frac{1}{2}(B^1)^2$)
To simplify, fix now $n = d = 2$.
Log-coordinates

$$(X_t, Y_t, A_t) = \left(B_t^1, B_t^2, \frac{1}{2}\int_0^t B_s^1 dB_s^2 - B_s^2 dB_s^1\right).$$

Group law

$$(x, y, a) * (x', y', a') = \left(x + x', y + y', a + a' + \frac{1}{2}(xy' - x'y)\right).$$

B has generator $\frac{1}{2}\Delta_G = \frac{1}{2}\left(X_1^2 + X_2^2\right)$ where in coordinates

$$X_1 = \partial_x - \frac{y}{2}\partial_a, \quad X_2 = \partial_y + \frac{x}{2}\partial_a.$$

The sub-laplacian Δ_G is hypoelliptic on $G_{n,d}$.

- Potential kernel :

$$\mu\hat{U}(h) := \int_{G_{d,n}} \mu(dg)u(g,h),$$

where u is the fundamental solution $\Delta_G u(\cdot,h) = -\delta_h$, and is given by

$$u(g,h) = N(g * h^{-1})^{-Q+2},$$

where $Q = Q(d,n)$ and N is an "homogeneous norm" on $G_{d,n}$. Special cases :

  - $n = 1$, $d \geqslant 3$: $Q = d$, N Euclidean norm on $\mathbb{R}^d$.
  - $n = d = 2$:

$$Q = 4, \quad N(x,y,a) = c_N \left(\left(x^2 + y^2 \right)^2 + 16a^2 \right)^{\frac{1}{4}}.$$

- Enhanced Brownian motion $\mathbf{B}$ satisfies the assumptions needed for existence and characterization of the Root barrier.

We need to find measures μ such that $\mu\hat{U} \leqslant \delta_0\hat{U}$ (if possible with a density which is easy to sample from) :

- Very explicit formulae (Gaveau '77, Bonfiglioli-Lanconelli '03) for exit measures for **B** from "sphere" $S_N(r) = \{g \in G_{d,n}, \quad N(g) = r\}$ $\rightarrow$ one can then take linear combinations of those measures ($\sim$ radial measures).

Proposition

Let $\widetilde{\mu}$ be any probability measure on $(0, \infty)$. Let

$$\mu = \int_0^\infty \widetilde{\mu}(dr) \frac{c}{r^{Q-1}} \int_{S_N(r)} \frac{|\nabla_{\mathcal{H}} N|^2 (g)}{|\nabla N| (g)} d\sigma(g),$$

where σ is the surface measure,and $|\nabla_{\mathcal{H}} N|^2 := \sum_{i=1}^n |X_i(N)|^2$. Then one has

$$\mu\hat{U} \leqslant \delta\hat{U}.$$

For instance, for $n = d = 2$, one can take

$$\mathrm{d}\mu\,(x, y, a) = \frac{4}{\pi} 1_{(x^2+y^2)^2+16a^2<1} \frac{x^2 + y^2}{\left((x^2 + y^2)^2 + 16a^2\right)^{1/2}} \mathrm{d}x\mathrm{d}y\mathrm{d}a.$$

Then $\mu\hat{U} \leqslant \delta_0\hat{U}$, and $\mu\hat{U}$ is bounded and continuous.

$\rightarrow$ one can then numerically compute $\mu\hat{U}$, and solve the obstacle problem

$$\begin{cases} \min[(\partial_t - \frac{1}{2}\Delta_G)f, f - \mu\hat{U}] = 0 & \text{on } (0, \infty) \times G, \\ f(0, \cdot) = \delta_0\hat{U} \end{cases}$$

to obtain the Root's barrier.

Note that due to symmetry, x, y will only appear through $x^2 + y^2$. So we only need to solve a PDE in 2 space dimensions (and then store a 2-dim array for barrier function r).

Work in progress : numerical implementation, application to simulation of SDEs with bounded increments (in time,space,area),...

Conclusion

Root's solution to the Skorokhod embedding :

- conceptually simple (just a hitting time)
- can be computed efficiently (obstacle problem, integral equation)
- many applications (model-free finance, numerics,...)