Lévy-Vasicek Models and the Long-Bond Return Process

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Pricing kernels

The Vasicek model is of course one of the oldest and most well studied models in the mathematical finance literature, and one might think that there is very little that is new that can be said about it.

But it turns out that there are some surprising features of the Vasicek model relating to the long rate of interest that are very suggestive when it comes to modelling long term interest rates in general.

In what follows we shall use a pricing kernel method.

This is not the way in which the Vasicek model is usually presented in the literature.

But we shall see that the pricing kernel formulation is the most effective for our purposes.

Let us recall first how pricing kernels work in the geometric Brownian motion (GBM) model for asset prices.

We fix a Brownian motion $\{W_t\}_{t\geq 0}$ on $(\Omega, \mathcal{F}, \mathbb{P})$, and take it to be adapted to a filtration $\{\mathcal{F}_t\}$.

$\mathbb{P}$ is taken to be the real-world measure.

The GBM model is characterised by the specification of a pricing kernel along with the price processes of one or more assets.

For the pricing kernel in this model we write

$$\pi_t = \mathrm{e}^{-rt}\,\mathrm{e}^{-\lambda W_t - \frac{1}{2}\lambda^2 t}, \tag{1}$$

where r is the interest rate, and $\lambda > 0$ is a risk aversion parameter.

If an asset pays no dividend over some connected interval of time, we require that the product of the pricing kernel π_t and the asset price S_t should be a $\mathbb{P}$-martingale over that interval.

If we take this martingale to be of the form

$$\pi_t S_t = S_0 \, \mathrm{e}^{\beta W_t - \frac{1}{2}\beta^2 t}, \tag{2}$$

we obtain

$$S_t = S_0 \, \mathrm{e}^{(r+\lambda\sigma)t} \, \mathrm{e}^{\sigma W_t - \frac{1}{2}\sigma^2 t}, \tag{3}$$

where $\sigma = \beta + \lambda$ is the volatility of the asset.

If the asset has a single payoff H_T at time T, and derives its value entirely from that payoff, then the value of the asset at time $t < T$ is given by

$$H_t = \frac{1}{\pi_t} \mathbb{E}_t[\pi_T H_T]. \tag{4}$$

In particular, if $H_T = 1$, then we recover the pricing formula for a discount bond, given by

$$P_{tT} = \frac{1}{\pi_t} \, \mathbb{E}_t[\pi_T]. \tag{5}$$

Vasicek pricing kernel

To generalize the GBM model, we shall now make the interest rate stochastic while keeping the risk aversion level constant.

In other words, we consider a pricing kernel of the form

$$\pi_t = \exp\left[-\int_0^t r_s \mathrm{d}s\right] \exp\left[-\lambda W_t - \frac{1}{2}\lambda^2 t\right]. \tag{6}$$

In the Vasicek model, the short rate process $\{r_t\}_{t\geq 0}$ is taken to be a mean-reverting Gaussian process of Ornstein-Uhlenbeck (OU) type, satisfying

$$\mathrm{d}r_t = k(\theta - r_t)\mathrm{d}t - \sigma \mathrm{d}W_t. \tag{7}$$

Here k, θ and σ are respectively the mean reversion rate, the mean reversion level, and the absolute volatility of the short rate.

The dynamical equation (7) can then be solved to give

$$r_t = \theta + (r_0 - \theta)\,\mathrm{e}^{-kt} - \sigma \int_0^t \mathrm{e}^{k(s-t)} \mathrm{d}W_s. \tag{8}$$

To write down the pricing kernel we need the integral of the short rate,

$$I_t = \int_0^t r_s \mathrm{d}s. \tag{9}$$

Substitution of (8) into (9) gives

$$I_t = \theta t + \frac{1}{k}\left(1 - \mathrm{e}^{-kt}\right)(r_0 - \theta) - \sigma \int_{s=0}^{t} \int_{u=0}^{s} \mathrm{e}^{k(u-s)} \mathrm{d}W_u \mathrm{d}s. \tag{10}$$

The double integral can be rearranged and reduced to give

$$I_t = \theta t + \frac{1}{k}\left(1 - \mathrm{e}^{-kt}\right)(r_0 - \theta) - \frac{\sigma}{k} \int_0^t \left(1 - \mathrm{e}^{k(u-t)}\right) \mathrm{d}W_u. \tag{11}$$

For some purposes it turns out to be useful to replace the stochastic integral with an expression involving the short rate to obtain

$$I_t = \theta t + \frac{1}{k}(r_0 - r_t) - \frac{\sigma}{k} W_t \,. \tag{12}$$

It follows that the Vasicek pricing kernel can be expressed in the form

$$\pi_t = \exp\left[-\left(\theta + \frac{1}{2}\lambda^2\right) t + \left(\frac{\sigma}{k} - \lambda\right) W_t - \frac{1}{k}(r_0 - r_t) \right]. \tag{13}$$

Derivation of the discount bond formula

We proceed to work out the price of a discount bond.

Recall that

$$P_{tT} = \frac{1}{\pi_t}\,\mathbb{E}_t[\pi_T]. \tag{14}$$

Begin by noting that π_T is equal to

$$\exp\left[-\left(\theta+\frac{1}{2}\lambda^2\right)T-\frac{1}{k}\left(1-\mathrm{e}^{-kT}\right)(r_0-\theta)+\int_0^T\left(\frac{\sigma}{k}-\lambda-\frac{\sigma}{k}\mathrm{e}^{k(u-T)}\right)\mathrm{d}W_u\right]$$

To get an expression for $\mathbb{E}_t[\pi_T]$, we need the identity

$$\mathbb{E}_t\exp\left[\int_t^T\left(\frac{\sigma}{k}-\lambda-\frac{\sigma}{k}\mathrm{e}^{k(u-T)}\right)\mathrm{d}W_u\right]=\exp\left[\frac{1}{2}\int_t^T\left(\frac{\sigma}{k}-\lambda-\frac{\sigma}{k}\mathrm{e}^{k(u-T)}\right)^2\mathrm{d}u\right].$$

We then obtain

$$\log P_{tT} = -\left(\theta+\frac{1}{2}\lambda^2\right)(T-t)-\frac{1}{k}\left(\mathrm{e}^{-kt}-\mathrm{e}^{-kT}\right)(r_0-\theta)$$
$$+\frac{1}{2}\int_t^T\left(\frac{\sigma}{k}-\lambda-\frac{\sigma}{k}\mathrm{e}^{k(u-T)}\right)^2\mathrm{d}u+\frac{\sigma}{k}\left(1-\mathrm{e}^{k(t-T)}\right)\int_0^t\mathrm{e}^{k(u-t)}\,\mathrm{d}W_u.$$

Earlier, we showed that

$$\sigma \int_0^t \mathrm{e}^{k(u-t)}\,\mathrm{d}W_u = \theta + (r_0 - \theta)\mathrm{e}^{-kt} - r_t. \tag{15}$$

After some further manipulations one obtains the following expression for the value of a T-maturity discount bond:

$$P_{tT} = \exp\left[-R_\infty(T-t) + \frac{1}{k}\left(1-\mathrm{e}^{k(t-T)}\right)(R_\infty - r_t) - \frac{1}{4}\frac{\sigma^2}{k^3}\left(1-\mathrm{e}^{k(t-T)}\right)^2\right]. \tag{16}$$

Here

$$R_\infty = \theta + \frac{\lambda\sigma}{k} - \frac{1}{2}\frac{\sigma^2}{k^2} \tag{17}$$

is the asymptotic bond yield, or exponential long rate of interest, defined by

$$R_\infty = -\lim_{T\to\infty}\frac{1}{T-t}\log P_{tT}. \tag{18}$$

Uniform integrability of the pricing kernel

We remark on the following feature of the Vasicek model, which doesn't seem to have been pointed out before.

The Vasicek model has the property that $R_\infty > 0$ if and only if the pricing kernel is uniformly integrable.

We recall that a collection $\mathcal{C}$ of random variables is said to be uniformly integrable (UI) if

$$\lim_{\delta\to\infty} \sup_{X\in\mathcal{C}} \mathbb{E}[|X|\mathbb{1}\{|X| > \delta\}] = 0. \quad (19)$$

A random process $\{X_t\}_{t\geq 0}$ is said to be UI if

$$\lim_{\delta\to\infty} \sup_{t} \mathbb{E}[|X_t|\,\mathbb{1}\{|X_t| > \delta\}] = 0. \quad (20)$$

For a pricing kernel we drop the absolute value sign, and the UI condition is

$$\lim_{\delta\to\infty} \sup_{t} \mathbb{E}[\pi_t\,\mathbb{1}\{\pi_t > \delta\}] = 0. \quad (21)$$

An alternative way of expressing this condition is as follows.

Let us introduce the natural numeraire process $\{n_t\}_{t\geq 0}$ (also known as the growth optimal portfolio or benchmark portfolio) defined by $n_t = 1/\pi_t$, and set $\kappa = 1/\delta$.

Then (21) becomes

$$\lim_{\kappa \to 0} \sup_t \mathbb{E}[\pi_t \, \mathbb{1}\{n_t < \kappa\}] = 0. \tag{22}$$

The expression $\mathbb{E}[\pi_t \, \mathbb{1}\{n_t < \kappa\}]$ is the price at time 0 of a digital put option on the natural numeraire with strike κ and maturity t.

Therefore the UI property is a condition on a family of option prices:

Proposition 1. *A pricing kernel is uniformly integrable if and only if for any fixed price level $\epsilon > 0$ there exists a strike $\kappa > 0$ such that the value of a digital put option on the natural numeraire is less than ϵ for all maturities.*

Uniform integrability in the Vasicek model

In the case of the Vasicek model, we have the following results.

Proposition 2. *If* $R_\infty > 0$ *then* $\{\pi_t\}$ *is uniformly integrable.*

This can be proven by applying the $\mathcal{L}^p$-test for uniform integrability.

The $\mathcal{L}^p$-test says that if a collection of random variables is bounded in $\mathcal{L}^p$ for some $p > 1$, then it is UI.

The result follows by verifying that there exists a constant $\gamma > 0$ and $p > 1$ such that $\mathbb{E}[\pi_t^p] < \gamma$ for all t.

Proposition 3. *If* $R_\infty < 0$ *then* $\{\pi_t\}$ *is not uniformly integrable.*

To see this, recall that if a collection of random variables is UI then it is bounded in $\mathcal{L}^1$.

Thus it suffices to show that if $R_\infty < 0$ then $\{\pi_t\}$ is not bounded in $\mathcal{L}^1$.

But note that $\mathbb{E}[\pi_t] = P_{0t}$ and that

$$P_{0t} = \exp\left[-R_\infty t + \frac{1}{k}\left(1 - \mathrm{e}^{-kt}\right)(R_\infty - r_0) - \frac{1}{4}\frac{\sigma^2}{k^3}\left(1 - \mathrm{e}^{-kt}\right)^2\right]. \qquad (23)$$

We see that if $R_\infty < 0$, this expression grows without bound as $t \to \infty$.

It remains to check what happens when $R_\infty = 0$.

Proposition 4. *If $R_\infty = 0$ then $\{\pi_t\}$ is not uniformly integrable.*

Here, the situation is a little more delicate.

The pricing kernel fails the $\mathcal{L}^p$ test, so we cannot conclude that it is UI.

But the pricing kernel is bounded in L^1, so we cannot conclude that it is *not* UI.

The conclusion is that we must go to the definition of uniform integrability and see directly what is going on.

Recall that uniform integrability is related to the prices of digital puts on the natural numeraire.

Specifically, we need to investigate $\lim_{\kappa\to 0}\sup_t \mathbb{E}[\pi_t \mathbb{1}\{n_t < \kappa\}]$.

Following a Black-Scholes type option price calculation, one finds that

$$\mathbb{E}[\pi_t \mathbb{1}(n_t < \kappa)] = \mathrm{e}^{A_t + \frac{1}{2}B_t^2} N\left[\frac{A_t + \frac{1}{2}B_t^2 + \log\kappa}{B_t}\right] \tag{24}$$

where $\{A_t\}$ and $\{B_t\}$ are deterministic functions given by

$$A_t = -\left(R_\infty + \frac{1}{2}\left(\frac{\sigma}{k} - \lambda\right)^2 t\right) + \frac{1}{k}\left(1 - \mathrm{e}^{-kt}\right)(\theta - r_0) \tag{25}$$

and

$$B_t^2 = \left(\frac{\sigma}{k} - \lambda\right)^2 t - 2\left(\frac{\sigma^2}{k^3} - \frac{\lambda\sigma}{k^2}\right)\left(1 - \mathrm{e}^{-kt}\right) + \frac{1}{2}\frac{\sigma^2}{k^3}\left(1 - \mathrm{e}^{-2kt}\right). \tag{26}$$

One can show that if $R_\infty = 0$ then

$$\lim_{\kappa\to 0}\sup_t \mathbb{E}[\pi_t \mathbb{1}\{n_t < \kappa\}] > 0 \tag{27}$$

and hence the pricing kernel is not UI in this case, as claimed.

The Lévy–Vasicek pricing kernel

The relation between the uniform integrability of the pricing kernel and the positivity of the long rate of interest in the Vasicek model leads us to conjecture that uniform integrability of the pricing kernel may be a condition that we want to impose on interest rate models in general.

As a step towards understanding what the general situation may be we propose to investigate a class of Lévy extensions of the Vasicek model.

In particular, building on work by Eberlein & Raible (1999) and Norberg (2004), and others, we use a Lévy process to drive the SDE for the short rate in the Vasicek model.

We develop the resulting Lévy–Vasicek models by use of pricing kernel methods.

We recall that in so-called geometric Lévy models (see, e.g., Brody, Hughston & Mackie 2012) with a constant value for the short rate r, the pricing kernel takes the form

$$\pi_t = \mathrm{e}^{-rt}\,\mathrm{e}^{-\lambda\xi_t - t\psi(-\lambda)}. \tag{28}$$

Here, $\{\xi_t\}$ is a Lévy process, and $\lambda > 0$ is a risk aversion parameter.

We assume that $\{\xi_t\}$ admits exponential moments.

In particular, we have

$$\mathbb{E}[\mathrm{e}^{\alpha\xi_t}] = \mathrm{e}^{\psi(\alpha)\,t} \tag{29}$$

for $\alpha \in A$ for some connected subset $A \subset \mathbb{R}$ containing the origin, and we require that $-\lambda \in A$.

We call $\psi(\alpha)$ the Lévy exponent.

In the Lévy–Vasicek model, we take the pricing kernel to be of the form

$$\pi_t = \exp\left[-\int_0^t r_s \mathrm{d}s - \lambda\xi_t - \psi(-\lambda)t\right], \tag{30}$$

where the short rate is assumed to be a Lévy-OU process satisfying a dynamical equation of the form

$$\mathrm{d}r_t = k(\theta - r_t)\mathrm{d}t - \sigma\,\mathrm{d}\xi_t. \tag{31}$$

The ensuing calculations follow closely the ones in the classical Vasicek model.

One finds that the pricing kernel in the Lévy-Vasicek model takes the form

$$\pi_t = \exp\left[-\big(\theta + \psi(-\lambda)\big)t + \left(\frac{\sigma}{k} - \lambda\right)\xi_t - \frac{1}{k}(r_0 - r_t)\right]. \tag{32}$$

The price of a discount bond is given by

$$P_{tT} = \exp\left[-\left(\theta + \psi(-\lambda)\right)(T - t) + \int_t^T \psi(\alpha_{uT})\,\mathrm{d}u + \frac{1}{k}\left(1 - \mathrm{e}^{k(t-T)}\right)(\theta - r_t)\right].$$

Here we have set

$$\alpha_{uT} = \frac{\sigma}{k} - \lambda - \frac{\sigma}{k}\,\mathrm{e}^{k(u-T)}. \tag{33}$$

A calculation then shows that the long rate of interest in the Lévy-Vasicek model is given by

$$R_\infty = -\lim_{T\to\infty} \frac{1}{T-t} \log P_{tT} = \theta + \psi(-\lambda) - \psi\left(\frac{\sigma}{k} - \lambda\right). \tag{34}$$

To gain some insight into the significance of this formula, it may be helpful to re-examine the case when the Lévy process is a Brownian motion.

In that case the Lévy exponent is given by $\psi(\alpha) = \frac{1}{2}\alpha^2$.

Thus we have

$$\psi(-\lambda) - \psi\left(\frac{\sigma}{k} - \lambda\right) = \frac{1}{2}\lambda^2 - \frac{1}{2}\left(\frac{\sigma}{k} - \lambda\right)^2 = \frac{\lambda\sigma}{k} - \frac{1}{2}\frac{\sigma^2}{k^2}. \tag{35}$$

Therefore,

$$R_\infty = \theta + \frac{\lambda\sigma}{k} - \frac{1}{2}\frac{\sigma^2}{k^2}, \tag{36}$$

in the Brownian case, as we found earlier.

We see that the two correction terms in the expression for the long rate arise as the difference of two Lévy exponents.

Now we can ask whether in Lévy-Vasicek models positivity of the long rate is associated with uniform integrability of the pricing kernel, as it is in the classical Vasicek model.

We are able to show the following result:

Proposition 5. *The pricing kernel of a Lévy-Vasicek model is UI if and only if* $R_\infty > 0$.

Thus the result we obtain for the Lévy-Vasicek model closely follows that of the classical Vasicek model.

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