Optimal Liquidation Trajectories for the Almgren-Chriss Model with Lévy Processes

Arne Lokka

(joint work with Junwei Xu)

London School of Economics

a.lokka@lse.ac.uk

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Overview

Introduction

The Almgren-Chriss model

Let Y_t denote the number of shares the investor holds at time t, which is assumed to take the form $Y_t = Y_0 - \int_0^t \xi_u \, du$. Then the investor sells at prices S given by

$$S_t = s + \sigma W_t + \alpha\big(Y_t - Y_0\big) - F(\xi_t), \qquad t \geq 0.$$

The temporary impact function F is of the form $F(x) = \beta x^{\gamma}$, where typically γ is around 0.6.

The objective for the investor is to maximize

$$\mathbb{E}\big[C_T^Y\big] - \lambda \mathrm{Var}\big(C_T^Y\big),$$

where C_T^Y is the cash position at time T corresponding to the liquidation strategy Y.

This talk

We consider an infinite time-horizon where the investor sells at prices given by

$$S_t = s + L_t + \alpha\big(Y_t - Y_0\big) - F(\xi_t), \qquad t \geq 0,$$

where L_t is a Lévy process and F is a function satisfying certain properties.

The objective for the investor is to find a strategy Y which maximize

$$\mathbb{E}\big[-\exp\big(-AC_\infty^Y\big)\big],$$

where A denotes the investor's risk aversion and C_∞^Y denotes the investor's cash position at the end of time.

Initial Observations

Observation 1

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Observation 3

Models based on Brownian motion underestimate the probability of large market movements.

Some Key Questions

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Question 4

Are there any natural relations between the temporary impact function F and the probability law of the underlying asset?

Model Description

Assumption

We assume that L is a non-trivial Lévy process such that L_1 has finite second moment and

$$\bar{\delta} = \inf\left\{\delta < 0 \mid \mathbb{E}\left[\mathrm{e}^{\delta L_1}\right] < \infty\right\} < 0.$$

Such a Lévy process admits a decomposition

$$L_t = \mu t + \sigma W_t + \int_{\mathbb{R}} x \, \widetilde{N}(t, dx), \quad t \geq 0,$$

where $\mu \in \mathbb{R}$ and $\sigma \geq 0$ are constants, W is a standard Brownian motion, $\widetilde{N}$ is a compensated Poisson random measure. We let ν denote the Lévy measure associated with L.

Model Description

Admissible strategies

Given an initial share position y, the set $\mathcal{A}(y)$ of admissible strategies consists of adapted, absolutely continuous, non-increasing processes Y of the form $Y_t = y - \int_0^t \xi_u \, du$, satisfying

$$\int_0^\infty \|Y_t\|_{L^\infty(\mathbb{P})} \, dt < \infty \qquad \text{if } \mu \neq 0, \tag{1}$$

and

$$\int_0^\infty \|Y_t\|^2_{L^\infty(\mathbb{P})} \, dt < \infty \qquad \text{if } \mu = 0. \tag{2}$$

We let $\mathcal{A}_D(y)$ denote the set of all deterministic strategies in $\mathcal{A}(y)$.

Model Description

The investor sells his shares at prices given by

$$S_t = s + L_t + \alpha\big(Y_t - Y_0\big) - F(\xi_t), \qquad t \geq 0,$$

where $\alpha \geq 0$ and $F : [0, \infty) \to [0, \infty)$ satisfies

(i) F is continuous and continuously differentiable on $(0, \infty)$;

(ii) $F(0) = 0$;

(iii) the function $x \mapsto xF(x)$ is strictly convex on $[0, \infty)$;

(iv) $\lim_{x \to 0} xF'(x)$ exists;

(v) the function $x \mapsto x^2 F'(x)$ is strictly increasing on $[0, \infty)$, and tends to $+\infty$ as $x \to \infty$.

The Cash Position

For an admissible liquidation strategy $Y \in \mathcal{A}(y)$, the investor's cash position at the end of time is

$$\begin{aligned} C_\infty^Y &= c - \int_0^\infty S_t \, dY_t \\ &= c + sy - \frac{1}{2}\alpha y^2 + \int_0^\infty Y_{t-} \, dL_t - \int_0^\infty \xi_t F(\xi_t) \, dt. \end{aligned}$$

Also, recall that the investor's optimisation problem is to find a $Y^* \in \mathcal{A}(y)$, which maximise

$$\mathbb{E}\big[-\exp\big(-AC_\infty^Y\big)\big].$$

Problem Simplification

It turns out that in order to solve the optimal liquidation problem, it is sufficient to solve (the derivation is similar to that of Schied, Schöneborn and Tehranchi (2010))

$$V(y) = \inf_{Y \in \mathcal{A}_D(y)} \int_0^\infty \Big\{ \kappa_A(Y_t) + A\xi_t F(\xi_t) \Big\} \, dt,$$

where

$$\kappa_A(y) = \ln \mathbb{E}\big[\exp\big(-AyL_1\big)\big].$$

Moreover, if $\mu > 0$, then there are no admissible optimal solutions. Hence from now on we assume that the asset has a drift $\mu \leq 0$.

The Optimal Liquidation Trajectory

Let $G : [0, \infty) \to [0, \infty)$ be the inverse function of $x \mapsto x^2 F'(x)$, and define

$$\tau = \int_0^y \frac{1}{G\left(\frac{\kappa_A(u)}{A}\right)}\, du, \qquad y \in [0, \bar{\delta}_A).$$

Let Y^* be the unique process satisfying

$$\int_{Y_t^*}^y \frac{1}{G\left(\frac{\kappa_A(u)}{A}\right)}\, du = t, \quad \text{if } t \le \tau, \qquad \text{and} \qquad Y_t^* = 0, \quad \text{if } t > \tau.$$

Then $Y^* \in \mathcal{A}_D(y)$ is the optimal liquidation strategy, and the associated speed ξ^* is given by

$$\xi_t^* = G\left(\frac{\kappa_A(Y_t^*)}{A}\right), \qquad 0 \le t < \tau.$$

Approximation of Exponential Models

Stock price data are typically calibrated to exponential Lévy models, so we want to derive a linear approximation of such models. Consider the exponential model

$$\widetilde{S}_t = s\exp\big(\widetilde{L}_t\big) + I_t,$$

where $I_t = \alpha(Y_t - Y_0) + F(\xi_t)$ is the price impact at time t, and $\widetilde{L}$ is a Lévy process with characteristics $(\widetilde{\mu}, \widetilde{\sigma}, \widetilde{\nu})$ satisfying $d\widetilde{\nu}(x) = \widetilde{f}(x)\,dx$ and $\int_{|z|\geq 1} \mathrm{e}^{2z}\,\widetilde{\nu}(dz) < \infty$. We want to choose a Lévy process L such that S given by

$$S_t = s + L_t + I_t$$

is the linear approximation of $\widetilde{S}$.

Approximation of Exponential Models

Solution

Define a measure ν by

$$\nu(dx) = \frac{1}{1+x}\,\widetilde{f}\Big(\ln(1+x)\Big)\,dx, \qquad x > -1,\, x \neq 0,$$

and set

$$m = \widetilde{\mu} + \frac{\widetilde{\sigma}^2}{2} + \int_{\mathbb{R}} (\mathrm{e}^z - 1 - z\mathbf{1}_{\{|z|<1\}})\,\widetilde{\nu}(dz).$$

Let L be a Lévy process with characteristics $(m, \widetilde{\sigma}, \nu)$. Then

$$S_t = s + sL_t + I_t$$

is the linear approximation of $\widetilde{S}$.

Example; Variance-Gamma

Consider the case where $\widetilde{L}$ is the variance-gamma process with parameters (θ, ρ, η). Then $\widetilde{\sigma} = 0$, and $\widetilde{\nu}$ has a density

$$\widetilde{f}(z) = \frac{1}{\eta|z|} \mathrm{e}^{Cz - D|z|}, \qquad z \in \mathbb{R},$$

where

$$C = \frac{\theta}{\rho^2} \qquad \text{and} \qquad D = \frac{\sqrt{\theta^2 + \frac{2\rho^2}{\eta}}}{\rho^2}.$$

Our assumptions are satisfied if $D - C > 2$, and the cumulant generating function $\widetilde{\kappa}$ admits the expression

$$\widetilde{\kappa}(x) = -\frac{1}{\eta} \ln\left(1 - \frac{x^2 \rho^2 \eta}{2} - \theta\eta x\right).$$

Example; Variance-Gamma

The Lévy measure for the process L appearing in the expression for the linear approximation of $\widetilde{S}$ in the VG case is

$$\nu(dx) = \begin{cases} \frac{-1}{\eta \ln(1+x)}(1+x)^{C+D-1}\,dx, & x \in (-1,0), \\ \frac{1}{\eta \ln(1+x)}(1+x)^{C-D-1}\,dx, & x \in (0,\infty). \end{cases}$$

The corresponding κ_A function is

$$\kappa_A^{vg}(u) = -Amu + \int_{-1}^{\infty} \left(\mathrm{e}^{-Aux} - 1 + Aux\right)\nu(dx),$$

where $m = \widetilde{\kappa}(1)$.

Examples; Power-Law Temporary Impact

In this case, the temporary impact function takes the form

$$F(x) = \beta x^{\gamma}.$$

For our numerical examples, we choose $\gamma = 0.6$ and $\beta = 4.7 \times 10^{-5}$ (which we believe are reasonable in view of Almgren et. al. (2005) when we work with a daily volatility of roughly 0.02, take daily volume to be 2×10^{6} and choose an initial stock price of $s = 100$). The function G then takes the form

$$G(x) = \left(\frac{x}{\beta\gamma}\right)^{\frac{1}{\gamma+1}}, \qquad x \geq 0,$$

and the optimal liquidation trajectory is given by

$$\xi_t^* = \left(\frac{\kappa_{sA}^{vg}(Y_t^*)}{A\beta\gamma}\right)^{\frac{1}{\gamma+1}}.$$

Example; Power-Law Temporary Impact

We choose a risk aversion of $A = 2$, and recall that the unaffected price process is given by $100 \times \exp\big(\widetilde{L}_t\big)$.

Exponential VG case

We choose $\widetilde{L}$ to be a VG Lévy process with parameters $\theta = -0.002$, $\rho = 0.02$ and $\eta = 0.6$, which are typical values for daily stock price data.

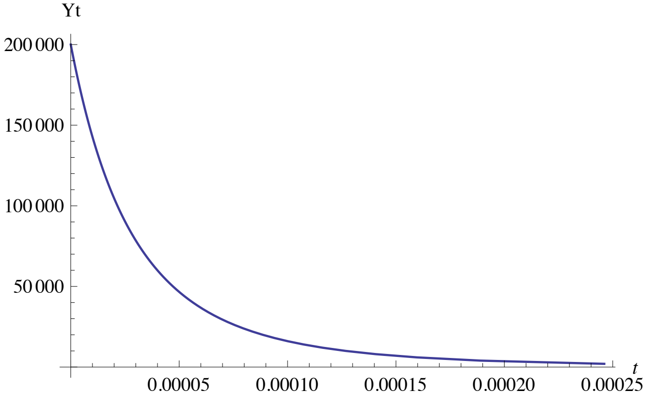
Exponential BM case

In this case we choose $\widetilde{L}_t = \mu t + \sigma W_t$, where W denotes a standard BM. We let the parameter μ and σ be given by $\mu + \frac{\sigma^2}{2} = \widetilde{\kappa}(1)$ and $2\mu + \frac{4\sigma^2}{2} = \widetilde{\kappa}(2)$, in which case the VG and the BM models have the same mean and variance.

Initial position

We assume that the investor initially holds $y = 2 \times 10^5$ number of shares, which is 10% of daily volume.

Optimal Liquidation Trajectory BM-Case



If the investor follows the optimal liquidation strategy, it takes roughly 2×10^{-4} days until the time the investor has liquidated 99% of his shares.

Optimal Liquidation Trajectory VG-Case

If the investor follows the optimal liquidation strategy, it takes less than 2×10^{-108263} days until the time he has liquidated 99% of his shares.

So essentially, the investor liquidates 99% of his shares instantly.

Even if we take $\gamma = 1000$, the time it takes the investor to liquidate 99% of a position of 2×10^5 number of shares would be less than 2×10^{-168}.

Optimal Liquidation Trajectory VG-Case

The previous example was an extreme case with risk aversion $A = 2$. Almgren typically assumes a risk aversion in the region $A = 10^4$ to $A = 10^5$.

For risk aversions in the region $A = 10^5$, the BM and the VG models will produce liquidation strategies that are comparable.

An Equivalence Relation

Consider the VG model with temporary impact function F_{vg} and the BM model with temporary impact function F_{bm}. Then the optimal liquidation trajectories for the two models coincide iff

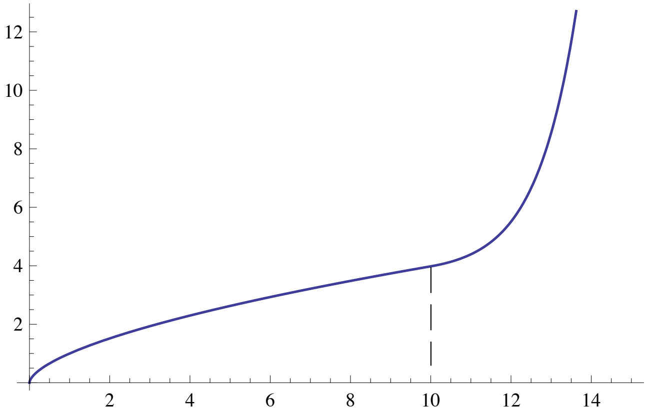
$$G_{vg}\left(\frac{\kappa_{sA}^{vg}(Y_t^*)}{A}\right) = \xi_t^* = G_{bm}\left(\frac{\kappa_{sA}^{bm}(Y_t^*)}{A}\right), \qquad t \geq 0,$$

where G_{vg} is the inverse function of $x \mapsto x^2 F'_{vg}(x)$ and G_{bm} is the inverse function of $x \mapsto x^2 F'_{bm}(x)$. We can solve for F_{vg} to obtain

$$F_{vg}(x) = \int_0^x \frac{1}{Az^2} \kappa_{sA}^{vg}\left(\left(\kappa_{sA}^{bm}\right)^{-1}\left(Az^2 F'_{bm}(z)\right)\right) dz.$$

If $F_{bm}(x) \sim x^\gamma$ for x small, then $F_{vg}(x) \sim x^\gamma$ for x small.

Temporary Impact; VG case



Conclusion

- We obtain semi-explicit solutions for the optimal liquidation trajectories when the risk is modelled by a Lévy process.
- We obtain an explicit expression for the connection between the temporary impact function for the Lévy model and the temporary impact function for the BM model, such that the optimal liquidation trajectories are identical.
- There might be a connection between the distribution of the returns and the temporary impact function, but this would require further investigations.

References

1 R. Almgren (2003), Optimal execution with nonlinear impact functions and trading-enhanced risk, *Risk* **10**, 1–18.

2 R. Almgren, C. Thum, E. Hauptmann and H. Li (2005), Direct estimation of equity market impact, *Risk* **18** (7), 58–62.

3 A. Schied, T. Schöneborn and M. Tehranchi (2010), Optimal basket liquidation for CARA investors is deterministic, *Applied Mathematical Finance* **17** (6), 471–489.

4 J. Kallsen and A. Shiryaev (2002), The cumulant process and Esscher's change of measure, *Finance and Stochastics* **6**, 397–428.

Thank You