Moral Hazard and mean field type interactions: A tale of a Principal and many Agents

Thibaut Mastrolia
CMAP, École Polytechnique

Joint work with Romuald Elie (Univ. Paris-Est Marne-La-Vallée) and Dylan Possamaï (Univ. Paris-Dauphine).

3rd London-Paris Bachelier workshop, Paris.

ÉCOLE POLYTECHNIQUE
UNIVERSITÉ PARIS-SACLAY

Motivations and general situation

Situation: A Principal takes the initiative of a contract which is proposed to an Agent. The Agent can accept or reject it (he is held to a given level).

Motivations and general situation

Situation: A Principal takes the initiative of a contract which is proposed to an Agent. The Agent can accept or reject it (he is held to a given level).

Problem: The Principal is potentially imperfectly informed about the actions of the Agent which impact her wealth (the output).

Motivations and general situation

Situation: A Principal takes the initiative of a contract which is proposed to an Agent. The Agent can accept or reject it (he is held to a given level).

Problem: The Principal is potentially imperfectly informed about the actions of the Agent which impact her wealth (the output).

Goal: Design a contract that maximises the utility of the Principal under constraints.

Moral hazard

The action of the Agent is hidden or not contractible.

Moral hazard

The action of the Agent is hidden or not contractible.

A Stackelberg-like equilibrium between the Principal and the Agent:

- compute the best-reaction function of the Agent given a contract
- determine his corresponding optimal effort

Moral hazard

The action of the Agent is hidden or not contractible.

A Stackelberg-like equilibrium between the Principal and the Agent:

- compute the best-reaction function of the Agent given a contract
- determine his corresponding optimal effort
- use this in the utility function of the Principal to maximise over all contracts.

The Holmström-Milgrom problem

Holmström-Milgrom (1985).

The Holmström-Milgrom problem

Holmström-Milgrom (1985).

- Fix a contract ξ. The Agent compute its best reaction effort given ξ. He solves (exponential utilities)

$$U_0^A(\xi) := \sup_{a \in \mathcal{A}} \mathbb{E}^{\mathbb{P}^a} \left[U_A \left(\xi - \int_0^T k(a_s) ds \right) \right]. \qquad (1)$$

Holmström-Milgrom (1985).

- Fix a contract ξ. The Agent compute its best reaction effort given ξ. He solves (exponential utilities)

$$U_0^A(\xi) := \sup_{a \in \mathcal{A}} \mathbb{E}^{\mathbb{P}^a}\left[U_A\left(\xi - \int_0^T k(a_s) ds \right) \right]. \quad (1)$$

- Martingale representation Theorem:

"(1) $\Longleftrightarrow$ solving a **Backward SDE** with a unique solution (Y, Z) ",

$$Y_t = \xi + \int_t^T \left(-\frac{R_A}{2} |Z_s|^2 + \sup_a \{ a_s Z_s - k(a_s) \} \right) ds - \int_t^T Z_s dB_s$$

The Holmström-Milgrom problem

Holmström-Milgrom (1985).

- Fix a contract ξ. The Agent compute its best reaction effort given ξ. He solves (exponential utilities)

$$U_0^A(\xi) := \sup_{a \in \mathcal{A}} \mathbb{E}^{\mathbb{P}^a}\left[U_A\left(\xi - \int_0^T k(a_s)ds\right)\right]. \tag{1}$$

- Martingale representation Theorem:

"(1) $\Longleftrightarrow$ solving a **Backward SDE** with a unique solution (Y, Z) ",

$$Y_t = \xi + \int_t^T \left(-\frac{R_A}{2}|Z_s|^2 + \sup_a\{a_s Z_s - k(a_s)\}\right) ds - \int_t^T Z_s dB_s$$

$$U_0^A(\xi) = -e^{-R_A Y_0}, \quad a^\star(Z).$$

The Holmström-Milgrom problem

Holmström-Milgrom (1985).

- Fix a contract ξ. The Agent compute its best reaction effort given ξ. He solves (exponential utilities)

$$U_0^A(\xi) := \sup_{a\in\mathcal{A}} \mathbb{E}^{\mathbb{P}^a}\left[U_A\left(\xi - \int_0^T k(a_s)ds\right)\right]. \qquad (1)$$

- Martingale representation Theorem:

"(1) $\Longleftrightarrow$ solving a **Backward SDE** with a unique solution (Y, Z) ",

$$Y_t = \xi + \int_t^T \left(-\frac{R_A}{2}|Z_s|^2 + \sup_a\{a_s Z_s - k(a_s)\}\right) ds - \int_t^T Z_s dB_s$$

$$U_0^A(\xi) = -e^{-R_A Y_0}, \quad a^\star(Z).$$

We get the following representation for admissible contract ξ

$$\xi = Y_0 - \int_0^T \left(-\frac{R_A}{2}|Z_s|^2 + \sup_a(aZ_s - k(a))\right) ds + \int_0^T Z_s dB_s.$$

The Principal's Problem:

$$U_0^P = \sup_{\xi,\ U_0^A(\xi) \geqslant R_0} \mathbb{E}^{\mathbb{P}^{a^\star(Z)}} \left[U_P(B_T - \xi) \right],$$

The Holmström-Milgrom problem and some extensions

The Principal's Problem:

$$U_0^P = \sup_{\xi,\ U_0^A(\xi) \geqslant R_0} \mathbb{E}^{\mathbb{P}^{a^\star(Z)}} \left[U_P(B_T - \xi) \right],$$

becomes

$$U_0^P = \sup_{Z,\ Y_0 \geqslant -\frac{ln(-R_0)}{R_A},} \mathbb{E}^{\mathbb{P}^{a^\star(Z)}} \left[U_P(B_T - \xi) \right].$$

The Holmström-Milgrom problem and some extensions

The Principal's Problem:

$$U_0^P = \sup_{\xi,\ U_0^A(\xi) \geqslant R_0} \mathbb{E}^{\mathbb{P}^{a^\star(Z)}} \left[U_P(B_T - \xi) \right],$$

becomes

$$U_0^P = \sup_{Z,\ Y_0 \geqslant -\frac{\ln(-R_0)}{R_A},} \mathbb{E}^{\mathbb{P}^{a^\star(Z)}} \left[U_P(B_T - \xi) \right].$$

- Holmström-Milgrom: continuous time settings. Extended then by Schättler and Sung; Sung; Müller; Hellwig and Schmidt ... see the book of Cvitanić and Zhang.
- Sannikov: continuous time payment and random retiring time,
- Cvitanić, Possamaï and Touzi: the Agent can control the volatility of the output: a dynamical approach of the problem.

The Holmström-Milgrom problem and some extensions

The Principal's Problem:

$$U_0^P = \sup_{\xi,\ U_0^A(\xi)\geqslant R_0} \mathbb{E}^{\mathbb{P}^{a^\star(Z)}}\left[U_P(B_T-\xi)\right],$$

becomes

$$U_0^P = \sup_{Z,\ Y_0\geqslant -\frac{ln(-R_0)}{R_A},} \mathbb{E}^{\mathbb{P}^{a^\star(Z)}}\left[U_P(B_T-\xi)\right].$$

- Holmström-Milgrom: continuous time settings. Extended then by Schättler and Sung; Sung; Müller; Hellwig and Schmidt ... see the book of Cvitanić and Zhang.
- Sannikov: continuous time payment and random retiring time,
- Cvitanić, Possamaï and Touzi: the Agent can control the volatility of the output: a dynamical approach of the problem.
- Some recent applications: Hajjej, Hillairet, Mnif and Pontier for Public Private Partnerships; Capponi and Frei: accidents prevention model. Among others...

The N-players model

Assume that the Principal can hire N-interacting Agents.

The N-players model

Assume that the Principal can hire N-interacting Agents.

Multi Agents models.

- **One period model**: Holmström; Mookherjee; Green and Stokey; Harris, Kriebel and Raviv; Nalebuff and Stiglitz (among others)
- **Continuous time**: Koo, Shim and Sung ; Elie and Possamaï.

The N-players model

Assume that the Principal can hire N-interacting Agents.

Multi Agents models.

- **One period model**: Holmström; Mookherjee; Green and Stokey; Harris, Kriebel and Raviv; Nalebuff and Stiglitz (among others)
- **Continuous time**: Koo, Shim and Sung ; Elie and Possamaï.

The Agents problems: We fix the contracts.

Find a Nash equilibrium $\Longleftrightarrow$ solve a multidimensional (qg-)BSDE

The N-players model

Assume that the Principal can hire N-interacting Agents.

Multi Agents models.

- **One period model**: Holmström; Mookherjee; Green and Stokey; Harris, Kriebel and Raviv; Nalebuff and Stiglitz (among others)
- **Continuous time**: Koo, Shim and Sung ; Elie and Possamaï.

The Agents problems: We fix the contracts.

Find a Nash equilibrium $\Longleftrightarrow$ solve a multidimensional (qg-)BSDE

- Not wellposed theory for a system of quadratic growth BSDEs (see Frei and Dos Reis for instance).
- Recent investigations: Xing and Žitković ; Harter and Richou.

The N-players model

Assume that the Principal can hire N-interacting Agents.

Multi Agents models.

- **One period model**: Holmström; Mookherjee; Green and Stokey; Harris, Kriebel and Raviv; Nalebuff and Stiglitz (among others)
- **Continuous time**: Koo, Shim and Sung ; Elie and Possamaï.

The Agents problems: We fix the contracts.

Find a Nash equilibrium $\Longleftrightarrow$ solve a multidimensional (qg-)BSDE

- Not wellposed theory for a system of quadratic growth BSDEs (see Frei and Dos Reis for instance).
- Recent investigations: Xing and Žitković ; Harter and Richou.
- Elie and Possamaï circumvents this problem by imposing wellposedness in the admissibility of the contracts.

The N-players model

Assume that the Principal can hire N-interacting Agents.

Multi Agents models.

- **One period model**: Holmström; Mookherjee; Green and Stokey; Harris, Kriebel and Raviv; Nalebuff and Stiglitz (among others)
- **Continuous time**: Koo, Shim and Sung ; Elie and Possamaï.

The Agents problems: We fix the contracts.

Find a Nash equilibrium $\Longleftrightarrow$ solve a multidimensional (qg-)BSDE

- Not wellposed theory for a system of quadratic growth BSDEs (see Frei and Dos Reis for instance).
- Recent investigations: Xing and Žitković ; Harter and Richou.
- Elie and Possamaï circumvents this problem by imposing wellposedness in the admissibility of the contracts.

The Principal problem: a standard stochastic control problem.
$2N$ state variables: the outputs controlled by the Agents and their continuation utilities.

The problem under interest

What happens when N goes to $+\infty$?

- Related to Mean Field Game theory. Introduced by Lasry and Lions; Huang, Caines and Malhamé.

The problem under interest

What happens when N goes to $+\infty$?

- Related to Mean Field Game theory. Introduced by Lasry and Lions; Huang, Caines and Malhamé.
- Typical situations: how a firm should provide electricity to a large population, how city planners should regulate a heavy traffic or a crowd of people.
- Systemic risk: study large number of banks and the underlying contagion phenomenon. See for instance Carmona, Fouque and Sun; Garnier, Papanicolaou and Yan; Fouque and Langsam...

The output process

Let $\Omega := \mathbb{R} \times \mathcal{C}([0,T];\mathbb{R})$, and $\mathbb{P}_0$ the Wiener measure. Fix a probability measure λ_0 (initial distribution of the state of the agent), $\mathbb{P} := \lambda_0 \otimes \mathbb{P}_0$. Canonical process (x, W). Let $\mathbb{F}$ be the completed natural filtration.

The output process

Let $\Omega := \mathbb{R} \times \mathcal{C}([0,T];\mathbb{R})$, and $\mathbb{P}_0$ the Wiener measure. Fix a probability measure λ_0 (initial distribution of the state of the agent), $\mathbb{P} := \lambda_0 \otimes \mathbb{P}_0$. Canonical process (x, W). Let $\mathbb{F}$ be the completed natural filtration.

$$X_t = x + \int_0^t \sigma_s(X) dW_s,\ \ t \in [0,T],\ \mathbb{P} - a.s.$$

The output process

Let $\Omega := \mathbb{R} \times \mathcal{C}([0,T];\mathbb{R})$, and $\mathbb{P}_0$ the Wiener measure. Fix a probability measure λ_0 (initial distribution of the state of the agent), $\mathbb{P} := \lambda_0 \otimes \mathbb{P}_0$. Canonical process (x, W). Let $\mathbb{F}$ be the completed natural filtration.

$$X_t = x + \int_0^t \sigma_s(X) dW_s, \ \ t \in [0,T], \ \mathbb{P} - a.s.$$

Let

- $\mu \in \mathcal{P}(\mathcal{C})$ "arbitrary distribution of the output managed by the infinitely many other Agents"

The output process

Let $\Omega := \mathbb{R} \times \mathcal{C}([0,T];\mathbb{R})$, and $\mathbb{P}_0$ the Wiener measure. Fix a probability measure λ_0 (initial distribution of the state of the agent), $\mathbb{P} := \lambda_0 \otimes \mathbb{P}_0$. Canonical process (x, W). Let $\mathbb{F}$ be the completed natural filtration.

$$X_t = x + \int_0^t \sigma_s(X) dW_s, \ t \in [0,T], \ \mathbb{P} - a.s.$$

Let

- $\mu \in \mathcal{P}(\mathcal{C})$ "arbitrary distribution of the output managed by the infinitely many other Agents"
- $q : [0,T] \longrightarrow \mathcal{P}(\mathbb{R})$ "arbitrary distribution of the actions chosen by these infinitely many Agents"

The output process

Let $\Omega := \mathbb{R} \times \mathcal{C}([0,T];\mathbb{R})$, and $\mathbb{P}_0$ the Wiener measure. Fix a probability measure λ_0 (initial distribution of the state of the agent), $\mathbb{P} := \lambda_0 \otimes \mathbb{P}_0$. Canonical process (x, W). Let $\mathbb{F}$ be the completed natural filtration.

$$X_t = x + \int_0^t \sigma_s(X) dW_s, \ \ t \in [0,T], \ \mathbb{P} - a.s.$$

Let

- $\mu \in \mathcal{P}(\mathcal{C})$ "arbitrary distribution of the output managed by the infinitely many other Agents"
- $q : [0,T] \longrightarrow \mathcal{P}(\mathbb{R})$ "arbitrary distribution of the actions chosen by these infinitely many Agents"
- $\alpha \in \mathcal{A}$, $\mathbb{F}$-adapted control process (+integrability conditions) for the representative Agent.

The output process

Let $\Omega := \mathbb{R} \times \mathcal{C}([0,T];\mathbb{R})$, and $\mathbb{P}_0$ the Wiener measure. Fix a probability measure λ_0 (initial distribution of the state of the agent), $\mathbb{P} := \lambda_0 \otimes \mathbb{P}_0$. Canonical process (x, W). Let $\mathbb{F}$ be the completed natural filtration.

$$X_t = x + \int_0^t \sigma_s(X) dW_s, \ t \in [0,T], \ \mathbb{P} - a.s.$$

Let

- $\mu \in \mathcal{P}(\mathcal{C})$ "arbitrary distribution of the output managed by the infinitely many other Agents"
- $q : [0,T] \longrightarrow \mathcal{P}(\mathbb{R})$ "arbitrary distribution of the actions chosen by these infinitely many Agents"
- $\alpha \in \mathcal{A}$, $\mathbb{F}$-adapted control process (+integrability conditions) for the representative Agent.

$$\frac{d\mathbb{P}^{\mu,q,\alpha}}{d\mathbb{P}} = \mathcal{E}\left(\int_0^T \sigma_t^{-1}(X) b(t, X, \mu, q_t, \alpha_t) dW_t \right).$$

$$\boxed{X_t = x + \int_0^t b(s, X, \mu, q_s, \alpha_s) ds + \int_0^t \sigma_s(X) dW_s^{\mu,q,\alpha}, \ t \in [0,T], \ \mathbb{P} - a.s.}$$

The Agent problem as an MFG problem

- Stackelberg equilibrium: For given ξ, and μ and q, the representative Agent has to solve

$$U_0^A(\mu,q,\xi) := \sup_{a\in\mathcal{A}} \underbrace{\mathbb{E}^{\mathbb{P}^{\mu,q,a}}\left[\xi - \int_0^T k_s(X,\mu,q_s,a_s)ds\right]}_{=:u_0^A(\mu,q,\xi,a)}.$$

The Agent problem as an MFG problem

- Stackelberg equilibrium: For given ξ, and μ and q, the representative Agent has to solve

$$U_0^A(\mu, q, \xi) := \sup_{a \in \mathcal{A}} \underbrace{\mathbb{E}^{\mathbb{P}^{\mu,q,a}} \left[\xi - \int_0^T k_s(X, \mu, q_s, a_s) ds \right]}_{=: u_0^A(\mu, q, \xi, a)} .$$

- Find a Mean field equilibrium: Lasry and Lions; Huang, Caines and Malhamé; Cardaliaguet; Bensoussan, Frehse and Yam; Guéant...

The Agent problem as an MFG problem

- Stackelberg equilibrium: For given ξ, and μ and q, the representative Agent has to solve

$$U_0^A(\mu, q, \xi) := \sup_{a \in \mathcal{A}} \underbrace{\mathbb{E}^{\mathbb{P}^{\mu,q,a}}\left[\xi - \int_0^T k_s(X, \mu, q_s, a_s)ds\right]}_{=: u_0^A(\mu,q,\xi,a)}.$$

- Find a Mean field equilibrium: Lasry and Lions; Huang, Caines and Malhamé; Cardaliaguet; Bensoussan, Frehse and Yam; Guéant...
- Solve the Mean Field Game problem: $(a^\star, \mu^\star, q^\star)$ such that

$$\textbf{(MFG)}(\xi) \begin{cases} u_0^A(\mu, q, \xi, a^\star) = U_0^A(\mu, q, \xi), \\ \mathbb{P}^{a^\star,\mu^\star,q^\star} \circ (X)^{-1} = \mu^\star \\ \mathbb{P}^{a^\star,\mu^\star,q^\star} \circ (a_t^\star)^{-1} = q_t^\star. \end{cases}$$

See the works of Carmona and Lacker; Lacker; Carmona, Delarue and Lacker...

The Agent problem: an other story of BSDEs

We now consider the following system which is intimately related to mean-field FBSDE

$$
\textbf{(MF-BSDE)}(\xi)\begin{cases} Y_t = \xi + \displaystyle\int_t^T \sup_\alpha \left(b(s,X,\mu,q_s,\alpha)Z_s - k_s(X,\mu,q_s,\alpha)\right)ds \\ \qquad - \int_t^T Z_s dX_s, \\ \mathbb{P}^{\alpha^\star(X,Z,\mu^\star,q^\star),\mu^\star,q^\star} \circ (X)^{-1} = \mu^\star, \\ \mathbb{P}^{\alpha^\star(X,Z,\mu^\star,q^\star),\mu^\star,q^\star} \circ (\alpha_t)^{-1} = q_t^\star. \end{cases}
$$

Similar studies on MF-BSDEs: Carmona and Delarue; Buckdahn, Djehiche, Li, and Peng; Li and Luo...

The Agent problem: an other story of BSDEs

$$\text{" Solve } (\textbf{MFG})(\xi) \Longleftrightarrow \text{ Solve } (\textbf{MF-BSDE})(\xi)\text{"}$$

The Agent problem: an other story of BSDEs

" Solve (**MFG**)(ξ) $\Longleftrightarrow$ Solve (**MF-BSDE**)(ξ)"

Theorem (Elie, M., Possamaï (16'))

- *Let ξ be such that* (**MFG**)(ξ) *admits a solution* $(\mu^\star, q^\star, a^\star)$. *Then there exists a solution* $(Y^\star, Z^\star, \mu^\star, q^\star)$ *to* (**MF-BSDE**)(ξ) *and* $a^\star$ *is a maximiser which provides an optimal effort. We thus have*

$$\xi = Y_0^\star - \int_0^T \left(b(s, X, \mu, q_s, a_s^\star) Z_s^\star - k_s(X, \mu, q_s, a_s^\star)\right) ds + \int_0^T Z_s^\star dX_s.$$

- *Conversely, if there exists a solution* $(Y^\star, Z^\star, \mu^\star, q^\star)$ *to* (**MF-BSDE**)(ξ) *then* (**MFG**)(ξ) *has a solution.*

The Agent problem: an other story of BSDEs

" Solve (**MFG**)(ξ) $\Longleftrightarrow$ Solve (**MF-BSDE**)(ξ)"

Theorem (Elie, M., Possamaï (16'))

- *Let ξ be such that* (**MFG**)(ξ) *admits a solution* $(\mu^\star, q^\star, a^\star)$. *Then there exists a solution* $(Y^\star, Z^\star, \mu^\star, q^\star)$ *to* (**MF-BSDE**)(ξ) *and* $a^\star$ *is a maximiser which provides an optimal effort. We thus have*

$$\xi = Y_0^\star - \int_0^T \left(b(s, X, \mu, q_s, a_s^\star) Z_s^\star - k_s(X, \mu, q_s, a_s^\star)\right) ds + \int_0^T Z_s^\star dX_s.$$

- *Conversely, if there exists a solution* $(Y^\star, Z^\star, \mu^\star, q^\star)$ *to* (**MF-BSDE**)(ξ) *then* (**MFG**)(ξ) *has a solution.*

Let us denote Ξ the set of admissible contracts ξ such that (**MFG**)(ξ) has a solution.

A fundamental characterization of Ξ

Let $Y_0 \in \mathbb{R}$ and Z predictable + integrability conditions. Let $\alpha^{\star,Z}$ be any maximiser of the generator of (**MF-BSDE**)(ξ). Consider the controlled McKean-Vlasov system:

$$
\textbf{(SDE)}_{MV}
\begin{cases}
X_t = x + \int_0^t b(s, X, \mu, q_s, \alpha_s^{\star,Z})ds + \int_0^t \sigma_s(X) dW_s^{\mu,q,\alpha^{\star,Z}_\cdot}, \\
Y_t^{Y_0,Z} = Y_0 + \int_0^t k_s(X, \mu, q_s, \alpha_s^{\star,Z})ds + \int_0^t Z_s \sigma_s(X) dW_s^{\mu,q,\alpha^{\star,Z}}, \\
\mu = \mathbb{P}^{\mu,q,\alpha^\star(\cdot,X,Z_\cdot,\mu,q_\cdot)} \circ X^{-1}, \\
q_t = \mathbb{P}^{\mu,q,\alpha^{\star,Z}} \circ (\alpha_t^{\star,Z})^{-1}.
\end{cases}
$$

Theorem (Elie, M., Possamaï (16'))

$$
\Xi = \left\{ Y_T^{Y_0,Z},\ Y_0 \geqslant R_0,\ Z \textit{ sufficiently integrable...} \right\}.
$$

The Principal problem: a non standard stochastic control problem

$$U_0^P := \sup_{\xi \in \Xi} \mathbb{E}^{\mathbb{P}^{\mu^\star, q^\star, \alpha^\star}} \left[X_T - \xi \right]$$

The Principal problem: a non standard stochastic control problem

$$U_0^P := \sup_{\xi \in \Xi} \mathbb{E}^{\mathbb{P}^{\mu^\star, q^\star, \alpha^\star}} [X_T - \xi] = \sup_{Y_0 \geqslant R_0} \sup_{Z} \mathbb{E}^{\mathbb{P}^{\mu^\star, q^\star, \alpha^\star}} \left[X_T - Y_T^{Y_0, Z} \right].$$

The Principal problem: a non standard stochastic control problem

$$U_0^P := \sup_{\xi \in \Xi} \mathbb{E}^{\mathbb{P}^{\mu^\star, q^\star, \alpha^\star}} [X_T - \xi] = \sup_{Y_0 \geqslant R_0} \sup_{Z} \mathbb{E}^{\mathbb{P}^{\mu^\star, q^\star, \alpha^\star}} \left[X_T - Y_T^{Y_0, Z} \right].$$

A stochastic optimal control problem with a two-dimensional state variable $M^Z := (X, Y^{Y_0, Z})$ controlled by Y_0 and Z. Two possible approaches:

The Principal problem: a non standard stochastic control problem

$$U_0^P := \sup_{\xi \in \Xi} \mathbb{E}^{\mathbb{P}^{\mu^\star, q^\star, \alpha^\star}} [X_T - \xi] = \sup_{Y_0 \geqslant R_0} \sup_{Z} \mathbb{E}^{\mathbb{P}^{\mu^\star, q^\star, \alpha^\star}} \left[X_T - Y_T^{Y_0, Z} \right].$$

A stochastic optimal control problem with a two-dimensional state variable $M^Z := (X, Y^{Y_0,Z})$ controlled by Y_0 and Z. Two possible approaches:

- Carmona and Delarue: using the maximum principle and the adjoint process of M^Z.
- Pham and Wei: using a dynamic programming principle and an **HJB equation** associated with the McKean-Vlasov optimal control problem **on the space of measures** (inspired by ideas of Lions).

The Principal problem: a non standard stochastic control problem

$$U_0^P := \sup_{\xi \in \Xi} \mathbb{E}^{\mathbb{P}^{\mu^\star, q^\star, \alpha^\star}} [X_T - \xi] = \sup_{Y_0 \geqslant R_0} \sup_{Z} \mathbb{E}^{\mathbb{P}^{\mu^\star, q^\star, \alpha^\star}} \left[X_T - Y_T^{Y_0, Z} \right].$$

A stochastic optimal control problem with a two-dimensional state variable $M^Z := (X, Y^{Y_0,Z})$ controlled by Y_0 and Z. Two possible approaches:

- Carmona and Delarue: using the maximum principle and the adjoint process of M^Z.
- Pham and Wei: using a dynamic programming principle and an **HJB equation** associated with the McKean-Vlasov optimal control problem **on the space of measures** (inspired by ideas of Lions).

On the admissibility of the contract (motivated by examples):

- Assume that the HJB equation has a solution with an optimal $z^\star$ for instance.
- We check that for this $z^\star$, the system $(\mathbf{SDE})_{MV}$ has indeed a solution and then $\xi^\star := Y_T^{R_0, z^\star}$ will be an optimal admissible contract.

Application: mean dependency and variance penalisation

$$b(s,x,\mu,q,a) := a + \alpha x + \beta_1 \int_{\mathbb{R}} x d\mu_s(x) + \beta_2 \int_{\mathbb{R}} x dq_s(x) - \gamma V_\mu(s),$$

$$V_\mu(s) := \int_{\mathbb{R}} |x|^2 d\mu_s(x) + \left| \int_{\mathbb{R}} x d\mu_s(x) \right|^2, \quad k(a) = \frac{a^2}{2}.$$

Application: mean dependency and variance penalisation

$$b(s,x,\mu,q,a) := a + \alpha x + \beta_1 \int_{\mathbb{R}} x d\mu_s(x) + \beta_2 \int_{\mathbb{R}} x dq_s(x) - \gamma V_\mu(s),$$

$$V_\mu(s) := \int_{\mathbb{R}} |x|^2 d\mu_s(x) + \left| \int_{\mathbb{R}} x d\mu_s(x) \right|^2, \quad k(a) = \frac{a^2}{2}.$$

Theorem (Elie, M. , Possamaï (16'))

The optimal contract for the problem of the Principal is

$$\xi^\star := \delta - \alpha(1+\beta_2) \int_0^T e^{(\alpha+\beta_1)(T-t)} X_t dt + (1+\beta_2) \int_0^T e^{(\alpha+\beta_1)(T-t)} dX_t,$$

for some constant δ explicitly given and the associated optimal effort of the Agent is

$$a_u^\star := (1+\beta_2) e^{(\alpha+\beta_1)(T-u)}, u \in [0,T].$$

Application: mean dependency and variance penalisation

$$b(s,x,\mu,q,a) := a + \alpha x + \beta_1 \int_{\mathbb{R}} x d\mu_s(x) + \beta_2 \int_{\mathbb{R}} x dq_s(x) - \gamma V_\mu(s),$$

$$V_\mu(s) := \int_{\mathbb{R}} |x|^2 d\mu_s(x) + \left| \int_{\mathbb{R}} x d\mu_s(x) \right|^2, \quad k(a) = \frac{a^2}{2}.$$

Theorem (Elie, M. , Possamaï (16'))

The optimal contract for the problem of the Principal is

$$\xi^\star := \delta - \alpha(1+\beta_2) \int_0^T e^{(\alpha+\beta_1)(T-t)} X_t dt + (1+\beta_2) \int_0^T e^{(\alpha+\beta_1)(T-t)} dX_t,$$

for some constant δ explicitly given and the associated optimal effort of the Agent is

$$a_u^\star := (1+\beta_2) e^{(\alpha+\beta_1)(T-u)}, \; u \in [0,T].$$

Extension:

$$U_0^P = \sup_\xi \mathbb{E}^{\mathbb{P}^\star}\Big[X_T - \xi\Big] - \lambda \mathrm{Var}_{\mathbb{P}^\star}(X_T) - \tilde{\lambda} \mathrm{Var}_{\mathbb{P}^\star}(\xi) + \hat{\lambda} \mathrm{Cov}_{\mathbb{P}^\star}(X_T, \xi).$$

Link with the N-agents model.

Let $(t, x, a) \in [0, T] \times \mathbb{R}^N \times A^N$,

$$b^N\big(t, x, \mu^N(x), a\big) := a + \alpha x + \beta_1 \int_{\mathbb{R}^N} w \mu^N(dw),$$

with $\mu^N(x)$ the empirical distribution of x.

Link with the N-agents model.

Let $(t, x, a) \in [0, T] \times \mathbb{R}^N \times A^N$,

$$b^N\big(t, x, \mu^N(x), a\big) := a + \alpha x + \beta_1 \int_{\mathbb{R}^N} w \mu^N(dw),$$

with $\mu^N(x)$ the empirical distribution of x.

Theorem (Elie, M., Possamaï (16'))

$$a_t^{N,\star} = \exp((\alpha + \beta_1)(T - t))\mathbf{1}_N.$$

In particular, the optimal effort of the ith Agent in the N players model coincides with the optimal effort of the Agent in the mean–field model. The optimal contract $\xi^{N,\star}$ proposed by the Principal is

$$\xi^{N,\star} = R_0^N - \int_0^T \frac{e^{2\kappa(T-t)}}{2} \mathbf{1}_N dt - \int_0^T e^{\kappa(T-t)} B_N X_t^N dt + \int_0^T e^{\kappa(T-t)} dX_t^N,$$

and for any $i \in \{1, \ldots, N\}$ we have

$$\mathbb{P}_N^{a^{N,\star}} \circ \left((\xi^{N,\star})^i\right)^{-1} \xrightarrow[N \to \infty]{\text{weakly}} \mathbb{P}^{a^\star} \circ (\xi^\star)^{-1}.$$

Thank you.