Convergence rate of strong approximations of compound random maps: an Application to Utility SPDEs

Mohamed MRAD
Joint work with Emmanuel GOBET

ÉcolePolytechnique, Université Paris 13,
emmanuel.gobet@polytechnique.edu, mrad@math.univ-paris13.fr

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Motivation: Stochastic PDEs

Consider a consistent stochastic utility process $\big(U(t,x,\omega)\big)_{t,x,\omega}$. It is shown in Progressive Stochastic Utilities [El Karoui,Mrad '13] : The consistent utilities solves a second-order fully nonlinear SPDE that can be solved by composition of stochastic flows.

- The consistent utilities solves a second-order fully nonlinear SPDE:

$$dU(t,x,\omega) = (-r_t x U_x + \frac{1}{2U_{xx}}||\gamma_x^{\mathcal{R}} + U_x \eta_t||^2)(t,x,\omega)dt + \gamma(t,x,\omega)dW_t \quad (1)$$

- We associate to this equation two SDEs: SDE(μ,κ) and SDE(b,ν).
- If these SDEs admit a strong solutions X and Y, with X monotonic. Then the utility SPDE can be solved by composition of stochastic flows, i.e. denoting by $\mathcal{X}$ the inverse flow of X:

$$U_x(t,x,\omega) = Y_t(u_x(0,\mathcal{X}_t(x,\omega)),\omega), \quad U(0,x,\omega) = u(0,x)$$

Questions.

♣ Can we find a numerical scheme to approximate U using this Connection between SPDE end two SDEs without discretizing the Dynamics of U (very complicated in practice)?

♣ Assume $\mathcal{X}^N$ converge to $\mathcal{X}$ with rate α^X and Y^N to Y with rate α^Y.

- Does $Y_t^N(u_x(0, \mathcal{X}_t^N))$ converges to $Y_t(u_x(0, \mathcal{X}_t))$?
- What is the convergence rate and how it depends on α^X and α^Y?

♣ If it is possible, can we extends this results to more general SPDEs?

General Setting

Consider

- $(\mathcal{E}, |.|)$ be a separable Banach space.
- $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space.
- a random field, i.e. a $\mathcal{F} \otimes \mathcal{B}(\mathbb{R}^d)$-measurable mapping $(\omega, x) \in (\Omega, \mathbb{R}^d) \mapsto F(\omega, x) \in \mathcal{E}$, continuous in x for a.e. ω;
- a $\mathcal{F}$-random variable $\Theta : \Omega \mapsto \mathbb{R}^d$.

Our Aim

- F^N and Θ^N are some approximations of F and Θ.
- Control in $\mathbf{L}_p$ the error $\omega \in \Omega \mapsto F^N(\omega, \Theta^N(\omega)) - F(\omega, \Theta(\omega)) \in \mathcal{E}$.
- Strong approximation rates: crucial for Multi-Level Monte Carlo methods.

♠ F and Θ may be dependent.

Definition.

♣ A random map $G(x, \omega)$ satisfies Assumption (H) if for any $p > 0$, $\exists \alpha_p, C_p \in [0, +\infty)$ s.t.

$$\left\| \sup_{|x| \leq \lambda} |G(\cdot, x)| \right\|_{L_p} \leq C_p^{(H)} \lambda^{\alpha_p^{(H)}}, \quad \forall \lambda \geq 1.$$

♣ A random map $H(x, y, \omega)$ satisfies Assumption (H') if $\exists \kappa \in (0, 1]$ s.t. $\forall p > 0, \exists \alpha_p, C_p \in [0, +\infty)$ s.t.

$$\left\| \sup_{x \neq y, |x| \leq \lambda, |y| \leq \lambda} |H(\cdot, x, y)| \right\|_{L_p} \leq C_p^{(H')} \lambda^{\alpha_p^{(H')}}, \quad \forall \lambda \geq 1.$$

Assumptions.

(H1) The random map F satisfies Assumption (H) with coefficients $C_p^{(\mathbf{H1})}$ and $\alpha_p^{(\mathbf{H1})}$.

(H2) $\exists \kappa \in (0,1]$ s.t. $\frac{|F(\cdot,y)-F(\cdot,x)|}{|y-x|^\kappa}$ satisfies (H') for some $C_p^{(\mathbf{H2})}$ and $\alpha_p^{(\mathbf{H2})}$.

(H3) The random map $F^N(\cdot,x) - F(\cdot,x)$ also satisfies (H) with coefficients $C_p^{N,(\mathbf{H3})}$ and $\alpha_p^{(\mathbf{H3})}$.

(H4) $\forall p > 0$, $\exists C_p^{(\mathbf{H4\text{-}a})}, (C_p^{N,(\mathbf{H4\text{-}b})})_{N\geq 1} \in [0,+\infty)$ s.t.

$$\|\Theta\|_{\mathrm{L}_p} \vee \left\|\Theta^N\right\|_{\mathrm{L}_p} \leq C_p^{(\mathbf{H4\text{-}a})}, \quad \forall N \geq 1, \tag{H4-a}$$

$$\left\|\Theta^N - \Theta\right\|_{\mathrm{L}_p} \leq C_p^{N,(\mathbf{H4\text{-}b})}, \quad \forall N \geq 1. \tag{H4-b}$$

Remarks.

♣ Had the random variable Θ be bounded by a finite constant Λ, we would have directly obtained $\left\|F^N(\Theta) - F(\Theta)\right\|_{\mathbf{L}_p} \leq C_p^{N,(\mathbf{H3})} \Lambda^{\alpha_p^{(\mathbf{H3})}}$.

♣ The extension to non bounded r.v. Θ is non trivial and is being achieved in our general Theorem and its proof.

♣ The following result is instrumental in our analysis. In particular, it enables to justify that the quantities of study are well defined as $\mathbf{L}_p$ random variables.

Intermediate Result:Inspired from [Kohatsu-Higa, A. and Sanz-SolÕ, M. (1997)]

Proposition.

Let G be a $\mathcal{F} \otimes \mathcal{B}(E)$-measurable mapping taking values in $\mathcal{E}$ satisfying (H) that is for any $p > 0$, $\exists \alpha_p^{(G)} \in [0, +\infty)$ and $C_p^{(G)} \in [0, +\infty)$ for which

$$\left\| \sup_{|x| \leq \lambda} |G(\cdot, x)| \right\|_{\mathbf{L}_p} \leq C_p^{(G)} \lambda^{\alpha_p^{(G)}}, \quad \forall \lambda \geq 1. \tag{2}$$

Let ξ be a random variable taking values in E, with finite $\mathbf{L}_p$ norms for any $p > 0$. Then for any $p > 0$, $\omega \mapsto G(\omega, \xi(\omega)) \in \mathbf{L}_p$ and for any finite conjugate exponents r and s $(r^{-1} + s^{-1} = 1)$, we have the estimate

$$\|G(\xi)\|_{\mathbf{L}_p} \leq C_{pr}^{(G)} \left(\zeta(r)\right)^{1/(pr)} 2^{\alpha_{pr}^{(G)} + 1/p} \left(1 + \|\xi\|_{\mathbf{L}_{s(\alpha_{pr}^{(G)} p + 1)}}^{\alpha_{pr}^{(G)} + 1/p}\right)$$

where $\zeta(r) := \sum_{n \geq 1} n^{-r}$ is the Riemann zeta function.

Remarks.

⇒ As a direct consequence of the above result, we deduce that $F(\Theta)$ is any $\mathbf{L}_p$ (owing to (**H1**) and (**H4**-a)).

⇒ Moreover we can also apply it to $G = F^N$ and $\xi = \Theta^N$ in view of (**H4**-a) and since (2) is satisfied (owing to (**H1**) and (**H3**)): Thus, $F^N(\Theta^N)$ also belongs to any $\mathbf{L}_p$.

Proof.

Using twice Hölder inequalities, we obtain

$$\begin{aligned}
\mathbb{E}\left(|G(\cdot,\xi)|^{p}\right) &\le \sum_{n\ge 1}\mathbb{E}\left(\sup_{|x|\le n}|G(\cdot,x)|^{p}1_{n-1\le|\xi|<n}\right)\\
&\le \sum_{n\ge 1}\left(\mathbb{E}\left(\sup_{|x|\le n}|G(\cdot,x)|^{pr}\right)\right)^{1/r}\mathbb{P}\left(n-1\le|\xi|<n\right)^{1/s}\\
&\le [C_{pr}^{(G)}]^{p}\sum_{n\ge 1}\frac{1}{n}\;n^{\alpha_{pr}^{(G)}p+1}\mathbb{P}\left(n-1\le|\xi|<n\right)^{1/s}\\
&\le [C_{pr}^{(G)}]^{p}\left(\sum_{n\ge 1}\frac{1}{n^{r}}\right)^{1/r}\left(\sum_{n\ge 1}n^{s(\alpha_{pr}^{(G)}p+1)}\mathbb{P}\left(n\le|\xi|+1<n+1\right)\right)^{1/s}\\
&\le [C_{pr}^{(G)}]^{p}\left(\zeta(r)\right)^{1/r}\left(\mathbb{E}\left((|\xi|+1)^{s(\alpha_{pr}^{(G)}p+1)}\right)\right)^{1/s}.
\end{aligned}$$

Therefore, $\|G(\xi)\|_{L_p} \le C_{pr}^{(G)}\left(\zeta(r)\right)^{1/(pr)}\left(1+\|\xi\|_{L_{s(\alpha_{pr}^{(G)}p+1)}}\right)^{\alpha_{pr}^{(G)}+1/p}$ where we have used the Minkowsky inequality. We complete our statement by using

$$(a+b)^{\gamma}\le 2^{(\gamma-1)_{+}}\left(a^{\gamma}+b^{\gamma}\right)\le 2^{\gamma}\left(a^{\gamma}+b^{\gamma}\right)$$

for any non-negative a, b, γ.

General Result

Theorem 1.

Assume **(H1)**-**(H2)**-**(H3)**-**(H4**-a)-**(H4**-b). Then for any $p > 0$ and any $p_2 > p$, there is a constant $c_{(1)}$ independent on N such that

$$\left\| F^N(\Theta^N) - F(\Theta) \right\|_{\mathbf{L}_p} \leq c_{(1)} \left(\underbrace{C_{2p}^{N,(\mathrm{H3})}}_{\|F^N(\theta)-F(\theta)\|_{\mathbf{L}_p}} + \underbrace{[C_{\kappa p_2}^{N,(\mathrm{H4\text{-}b})}]^{\kappa}}_{[\|\Theta^N-\Theta\|_{\mathbf{L}_{p_2}}]^{\kappa}} \right), \forall N \geq 1.$$

Quite intuitively, the global approximation error inherits from that on F and that on Θ modified by the local Hölder regularity of $x \mapsto F(\omega, x)$.

Corollary.

- $F^N - F$" $=$ " $O(N^{-\gamma_F})$ in any L_p
- $\Theta^N - \Theta$" $=$ " $O(N^{-\gamma_\Theta})$ in any L_p

The order of L_p-convergence of $F^N(\Theta^N) - F(\Theta)$ is $\min(\gamma_F, \kappa\gamma_\Theta)$.

Proof (I).

Write $F^N(\Theta^N) - F(\Theta) = \left[F^N(\Theta^N) - F(\Theta^N)\right] + \left[F(\Theta^N) - F(\Theta)\right]$. First, a direct application of previous Proposition (for $r = s = 2$) with **(H3)** and **(H4**-a) yields

$$\left\|F^N(\Theta^N) - F(\Theta^N)\right\|_{\mathrm{L}_p} \le C_{2p}^{N,(\mathrm{H3})} (\zeta(2))^{1/(2p)} 2^{\alpha_{2p}^{(\mathrm{H3})}+1/p} \left(1 + \left\|\Theta^N\right\|_{\mathrm{L}_{2(\alpha_{2p}^{(\mathrm{H3})}p+1)}}^{\alpha_{2p}^{(\mathrm{H3})}+1/p}\right)$$

$$\le C_{2p}^{N,(\mathrm{H3})} (\zeta(2))^{1/(2p)} 2^{\alpha_{2p}^{(\mathrm{H3})}+1/p} \left(1 + [C_{2(\alpha_{2p}^{(\mathrm{H3})}p+1)}^{(\mathrm{H4\text{-}a})}]^{\alpha_{2p}^{(\mathrm{H3})}+1/p}\right).$$

Consider now the second term $F(\Theta^N) - F(\Theta)$: Set

$$H_\kappa(\omega, \lambda) := \sup_{x \neq y, |x| \le \lambda, |y| \le \lambda} \frac{|F(\omega, y) - F(\omega, x)|}{|y - x|^\kappa}$$

and write $|F(\Theta^N) - F(\Theta)| \le H_\kappa(|\Theta^N| \vee |\Theta|)|\Theta^N - \Theta|^\kappa$. Then the Hölder inequality with p-conjugate numbers (p_1, p_2) (i.e. $p_1^{-1} + p_2^{-1} = p^{-1}$) gives

$$\left\|F(\Theta^N) - F(\Theta)\right\|_{\mathrm{L}_p} \le \left\|H_\kappa(|\Theta^N| \vee |\Theta|)\right\|_{\mathrm{L}_{p_1}} \left\|\Theta^N - \Theta\right\|_{\mathrm{L}_{\kappa p_2}}^{\kappa}.$$

Proof (II).

The first factor is upper bound using previous Proposition (for $r = s = 2$) with (**H2**) and (**H4**-b), it readily leads to

$$
\begin{aligned}
&\left\| F(\Theta^N) - F(\Theta) \right\|_{\mathbf{L}_p} \\
&\leq \left\| H_\kappa(|\Theta^N| \vee |\Theta|) \right\|_{\mathbf{L}_{p_1}} \left\| \Theta^N - \Theta \right\|^{\kappa}_{\mathbf{L}_{\kappa p_2}} \\
&\leq C^{(\mathbf{H2})}_{2p_1} (\zeta(2))^{1/(2p_1)} \, 2^{\alpha^{(\mathbf{H2})}_{2p_1} + 1/p_1} \left(1 + \left\| |\Theta^N| \vee |\Theta| \right\|^{\alpha^{(\mathbf{H2})}_{2p_1} + 1/p_1}_{\mathbf{L}_{2(\alpha^{(\mathbf{H2})}_{2p_1} p_1 + 1)}} \right) [C^{N,(\mathbf{H4}\text{-b})}_{\kappa p_2}]^{\kappa} \\
&\leq C^{(\mathbf{H2})}_{2p_1} (\zeta(2))^{1/(2p_1)} \, 2^{\alpha^{(\mathbf{H2})}_{2p_1} + 1/p_1} \left(1 + [2C^{(\mathbf{H4}\text{-a})}_{2(\alpha^{(\mathbf{H2})}_{2p_1} p_1 + 1)}]^{\alpha^{(\mathbf{H2})}_{2p_1} + 1/p_1} \right) [C^{N,(\mathbf{H4}\text{-b})}_{\kappa p_2}]^{\kappa}.
\end{aligned}
$$

We are done.

(Simplified Assumptions).

In some situations, checking the assumptions (**H1**-**H2**-**H3**) may be difficult since we evaluate the $\mathbf{L}_p$-norms of a maximum.

- When x is a time variable, we may rely on Doob inequalities and other martingale estimates to achieve this.
- In other situations, it becomes much more complicated. One can apply the general Kolmogorov continuity criterion for random fields (see Theorem 1.4.1 p.31 of the refernce book of H. Kunita), but it does not yield the quantitative estimates we are looking for, in particular regarding the polynomial growth factor in (**H1**-**H2**-**H3**).

Alternatively, here we use the Garsia-Rodemich-Rumsey lemma which gives refinement compared to the Kolmogorov criterion.

Assumptions: How To Get Uniform Estimates From Local Ones

Proposition (Garsia-Rodemich-Rumsey, control of modulus of continuity).

Let ρ, $\Psi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ be continuous and strictly increasing functions vanishing at zero and such that $\lim_{t\to+\infty} \Psi(t) = +\infty$. Suppose that $\phi : \mathbb{R}^d \longrightarrow \mathcal{E}$ is a continuous function with values on the separable Banach space $(\mathcal{E}, |.|)$. Denote by B_r the open ball in $\mathbb{R}^d$ centered at 0 with radius r. Then, provided

$$\Gamma = \int_{\mathrm{B}_r} \int_{\mathrm{B}_r} \Psi\Big(\frac{|\phi(x) - \phi(y)|}{\rho(|x-y|)}\Big) \mathrm{d}x \, \mathrm{d}y < +\infty$$

it holds, for all $x, y \in \mathrm{B}_r$,

$$|\phi(x) - \phi(y)| \leq 8 \int_0^{2|x-y|} \Psi^{-1}\Big(\frac{4^{d+1}\Gamma}{\lambda_d u^{2d}}\Big) \rho(\mathrm{d}u)$$

where λ_d is a universal constant depending only on d.

Assumptions: How To Get Uniform Estimates From Local Ones

Proposition.

Let $p > d$. Let G is $\mathcal{F} \otimes \mathcal{B}(\mathbb{R}^d)$- measurable mapping $(\omega, x) \in \Omega \times \mathbb{R}^d \mapsto G(,\omega, x) \in \mathcal{E}$, continuous in x for a.e. ω s.t. $G(x)$ is in L_p for any x and there exist constants $C^{(G)} \in [0, +\infty)$, $\beta^{(G)} \in (d/p, 1]$ and $\tau^{(G)} \in [0, +\infty)$ satisfying

$$\|G(x) - G(y)\|_{\mathbf{L_p}} \leq C^{(G)}|x-y|^{\beta^{(G)}}(1+|x|+|y|)^{\tau^{(G)}}, \quad \forall (x,y) \in \mathbb{R}^d \times \mathbb{R}^d.$$

Then, **(H1)** and **(H2)** holds true: i.e., for any $\beta \in (0, \beta^{(G)} - d/p)$,

$$\sup_{\lambda \geq 1} \lambda^{\beta - \tau^{(G)} - \beta^{(G)}} \left\| \sup_{x \neq y, |x| \leq \lambda, |y| \leq \lambda} \frac{|G(y) - G(x)|}{|y-x|^{\beta}} \right\|_{\mathbf{L_p}} < +\infty,$$

$$\sup_{\lambda \geq 1} \lambda^{-\tau^{(G)} - \beta^{(G)}} \left\| \sup_{|x| \leq \lambda} |G(x)| \right\|_{\mathbf{L_p}} < +\infty,$$

AN APPLICATION TO COMPOUND SDEs

♣ Standard filtered probability space $(\Omega, \mathbb{F}, \mathbb{P})$ supporting two q-dimensional standard Brownian motions $W = (W^1, \ldots, W^q)$ and $B = (B^1, \ldots, B^q)$ on $[0, T]$. W and B may be dependent.

♣ Two $\mathbb{R}^d$-valued stochastic processes X and Y, solutions of the SDEs (with Lipschitz coefficients)

$$\mathrm{d}X_t(x) = \mu(t, X_t(x))\mathrm{d}t + \sum_{i=1}^{q} \sigma_i(t, X_t(x))\mathrm{d}W_t^i, \quad X_0(x) = x,$$

$$\mathrm{d}Y_t(y) = b(t, Y_t(y))\mathrm{d}t + \sum_{i=1}^{q} \gamma_i(t, Y_t(y))\mathrm{d}B_t^i, \quad Y_0(y) = y,$$

♣ Denote by $X_T^N(x)$ (resp. $Y_T^N(y)$) the related Euler scheme with time step T/N of $X_T(x)$ (resp. $Y_T(y)$).

Aim: Approximation of $X_t(Y_t(y))$ by $X_t^N(Y_t^N(y)), t \in [0, T]$

Assumptions (For X).

(HP1) The coefficients μ and σ are Lipschitz continuous in space uniformly in time. $\exists C^{\mathbf{X}}$ s.t. $\forall t \in [0,T]$ and $x, y \in \mathbb{R}^{\mathbf{d}}$

$$\begin{cases} |\mu(t,x) - \mu(t,y)| \le C^{\mathbf{X}}|x-y|, & |\mu(t,0)| \le C^{\mathbf{X}}, \\ |\sigma(t,x) - \sigma(t,y)| \le C^{\mathbf{X}}|x-y|, & |\sigma(t,0)| \le C^{\mathbf{X}}. \end{cases} \quad \textbf{(HP1)}$$

(HP2) μ and σ are continuously space-differentiable functions such that their derivatives are δ-Hölder for a certain exponent $\delta \in (0,1]$.

$$\begin{cases} |\nabla_{\mathbf{x}}\mu(t,x) - \nabla_{\mathbf{x}}\mu(t,y)| \le C^{\mathbf{X},\nabla}|x-y|^{\delta}, & |\nabla_{\mathbf{x}}\mu(t,x)| \le C^{\mathbf{X},\nabla}, \\ |\nabla_{\mathbf{x}}\sigma(t,x) - \nabla_{\mathbf{x}}\sigma(t,y)| \le C^{\mathbf{X},\nabla}|x-y|^{\delta}, & |\nabla_{\mathbf{x}}\sigma(t,x)| \le C^{\mathbf{X},\nabla}. \end{cases} \quad \textbf{(HP2)}$$

(HP3) μ and σ are $\alpha^{\mathbf{X}}$-Hölder continuous in time, locally in space,

$$|\mu(t,x) - \mu(s,x)| + |\sigma(t,x) - \sigma(s,x)| \le C^{\mathbf{X}}(1+|x|)|t-s|^{\alpha^{\mathbf{X}}}. \quad \textbf{(HP3)}$$

(HP4) μ and σ are continuously space-differentiable functions such that their derivatives are $\alpha^{\mathbf{X}}$-Hölder continuous in time, locally in space,

$$|\nabla_{\mathbf{x}}\mu(t,x) - \nabla_{\mathbf{x}}\mu(s,x)| + |\nabla_{\mathbf{x}}\sigma(t,x) - \nabla_{\mathbf{x}}\sigma(s,x)| \le C^{\mathbf{X},\nabla}(1+|x|)|t-s|^{\alpha^{\mathbf{X}}}. \quad \textbf{(HP4)}$$

Assumptions (For Y).

(**HP1**) and (**HP3**) are satisfied for b and γ (instead of μ and σ) with a Hölder coefficient α^{Y} (instead of α^{X}).

Theorem 2.

The compound Euler scheme $X^N_{\cdot}(Y^N_{\cdot})$ converges to $X_{\cdot}(Y_{\cdot})$ in any $\mathbf{L}_p$-norm, at the order $\beta := \min(\alpha^X, \alpha^Y, \frac{1}{2})$ w.r.t. N: For any $p > 0$, there is a finite constant C_p such that for any $t \in [0, T]$

$$\sup_{t\in[0,T]} \left\| X^N_t(Y^N_t) - X_t(Y_t) \right\|_{\mathbf{L}_p} \leq C_p N^{-\beta}, \quad \forall N \geq 1.$$

SKETCH OF PROOF

Proposition.

Assume (**HP1**). For any $p > 0$, $\exists C_{p,(3)}$ and $C_{p,(4)}$ s.t.

$$\|X_t(x)\|_{L_p} \leq C_{p,(3)}(1 + |x|), \tag{3}$$

$$\|X_t(x) - X_t(y)\|_{L_p} \leq C_{p,(4)}|x - y| \tag{4}$$

for any $(t, x, y) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}^d$.

Corollary (From local to uniform estimates).

Assume Assumption (**HP1**). For any $p > 0$ and any $\beta \in (0, 1)$, there exist generic constants $C_{p,(5)}$ and $C_{p,(6)}$ such that, for any $t \in [0, T]$,

$$\sup_{t \in [0,T]} \left\| \sup_{|x| \leq \lambda} |X_t(x)| \right\|_{L_p} \leq C_{p,(5)}\lambda, \quad \forall \lambda \geq 1, \tag{5}$$

$$\sup_{t \in [0,T]} \left\| \sup_{x \neq y, |x| \leq \lambda, |y| \leq \lambda} \frac{|X_t(x) - X_t(y)|}{|y - x|^\beta} \right\|_{L_p} \leq C_{p,(6)}\lambda^{1-\beta}, \quad \forall \lambda \geq 1. \tag{6}$$

These estimates are also valid for X^N.

SKETCH OF PROOF

Proposition.

Assume (**HP2**). For any $p > 0$ $\exists C_{p,(7)}$ and $C_{p,(8)}$ s.t.

$$\|\nabla X_t(x)\|_{L_p} \leq C_{p,(7)}, \tag{7}$$

$$\|\nabla X_t(x) - \nabla X_t(y)\|_{L_p} \leq C_{p,(8)}|x-y|^{\delta} \tag{8}$$

for any $(t,x,y) \in [0,T] \times \mathbb{R}^d \times \mathbb{R}^d$.

Corollary (From local to uniform estimates).

Assume (**HP1**) and (**HP2**). For any $p > 0$ and any $\beta \in (0,\delta)$, $\exists C_{p,(9)}$, $C_{p,(10)}$ and $C_{p,(11)}$ s.t.,

$$\sup_{t\in[0,T]} \left\| \sup_{|x|\leq\lambda} |\nabla X_t(x)| \right\|_{L_p} \leq C_{p,(9)}\lambda^{\delta}, \quad \forall \lambda \geq 1, \tag{9}$$

$$\sup_{t\in[0,T]} \left\| \sup_{x\neq y, |x|\leq\lambda, |y|\leq\lambda} \frac{|\nabla X_t(x) - \nabla X_t(y)|}{|y-x|^{\beta}} \right\|_{L_p} \leq C_{p,(10)}\lambda^{\delta-\beta}, \quad \forall \lambda \geq 1, \tag{10}$$

$$\sup_{t\in[0,T]} \left\| \sup_{x\neq y, |x|\leq\lambda, |y|\leq\lambda} \frac{|X_t(x) - X_t(y)|}{|y-x|} \right\|_{L_p} \leq C_{p,(11)}\lambda^{\delta}, \quad \forall \lambda \geq 1. \tag{11}$$

SKETCH OF PROOF

To obtain locally uniform in space convergence results, the supplementary assumptions of regularity in space and time for $\nabla_x \mu$ and $\nabla_x \sigma_i$ (see (**HP2**) and (**HP4**)) are seemingly important. Thus classical strong convergence Theorem can be generalized to the following crucial one.

Theorem 3 (Theorem Strong convergence (new results)).

Assume (**HP1**), (**HP2**), (**HP3**), (**HP4**) and let $\beta = \min(\alpha, \frac{1}{2})$. For any $p > 0$, there exists a generic constant $C_{p,(12)}$ such that

$$\left\| \sup_{u \leq t} |X_u(x) - X_u^N(x) - X_u(y) + X_u^N(y)| \right\|_{L_p} \leq C_{p,(12)} (1+|x|+|y|) \frac{|x-y| + |x-y|^{\delta}}{N^{\beta}} \tag{12}$$

for all $x, y \in \mathbb{R}^d$ and $t \in [0, T]$.

We can now derive estimates locally uniformly in space, using the Garsia-Rodemich-Rumsey lemma applied to $G(x) = X_t(x) - X_t^N(x)$.

SKETCH OF PROOF

Theorem 4.

Under Assumptions (**HP1**)-(**HP4**), for any $p > 0$ there exists a finite generic constant $C_{p,(13)}$ such that, for any $t \in [0, T]$,

$$\sup_{t\in[0,T]} \left\| \sup_{|x|\leq\lambda} |X_t(x) - X_t^N(x)| \right\|_{\mathbf{L}_p} \leq \frac{C_{p,(13)}}{N^\beta}\lambda^2, \quad \forall\lambda \geq 1. \qquad (13)$$

The rest of the proof of $\sup_{t\in[0,T]} \left\|X_t^N(Y_t^N) - X_t(Y_t)\right\|_{\mathbf{L}_p} = O(N^{-\beta})$ follows by applying the main theorem.

Comeback to The Utility SPDE

Let b and γ regular enough and $Y_{s,t}(y), s \leq t \in [0, T]$ the strong solution of

$$dY_{s,t}(y) = b(t, Y_{s,t}(y))dt + \gamma(t, Y_{s,t}(y))dW_t, \;\; Y_{s,s}(y) = y$$

They are four approaches for computing the inverse flow: $\xi_{s,t}(y) := (Y_{s,t})^{-1}(y)$

i) as an inverse of a (random) function;
ii) as a forward in time SPDE;
iii) as a forward in time SDE with stochastic coefficients;
iv) as a backward in time SDE with standard coefficients.

i) Inverse flow as an inverse of a function: This means that we seek to directly invert the function $y \mapsto Y_t(s, y)$ without exploiting its stochastic dynamics. As such, we can use a dichotomy method or a Newton method: this would essentially require to compute a sequence of Euler schemes of $Y_t(s, y)$ for different y. This can be performed but we look for a less expensive scheme.

ii) Inverse flow as a forward in time SPDE: We recall the dynamics of $\xi_{s,t}(x)$ in the forward variable t, see [H. Kunita '97, Theorem 4.4.2 p.148] or [El Karoui & Mrad '13, Theorems 2.5 and 2.6].

Lemma.

Assume, in addition to Assumption (**HP1**), that b and γ are of class C^3 with bounded derivatives such that $\partial_x^3 b$ and $\partial_x^3 \gamma$ are δ-Hölder ($\delta > 0$). Then, for any given s, the inverse flow ξ is a semimartingale with respect to t and evolves as

$$\mathrm{d}\xi_{s,t}(x) = -\partial_x \xi_{s,t}(x) \Big[\big[b(t,x) - \partial_x \gamma(t,x) \cdot \gamma(t,x) \big] dt + \gamma(t,x) \cdot \mathrm{d}B_t \Big] + \frac{1}{2} \partial_{xx} \xi_{s,t}(x) |\gamma(t,x)|^2 \mathrm{d}t. \quad (14)$$

Certainly this SPDE is simpler than (1), but its discretization gives rise to delicate issues.

- We have inevitably to approximate $\partial_x \xi_{s,t}$ and $\partial_{xx} \xi_{s,t}$, using a finite differences method for example, which requires the resolution in the full space (or on a grid in x). This is computationally demanding.
- In addition, it seems really difficult to obtain error estimates in that context.

iii) Inverse flow as a forward in time SDE with stochastic coefficients: Alternatively, we may re-interpret $\xi_{s,t}$ as a SDE, in order to be a position to use existing approximation schemes (like Euler schemes). Namely, using

$$\partial_x \xi_{s,t}(x) = \frac{1}{(\partial_y Y_{s,t})(\xi_{s,t}(x))}, \qquad \partial_{xx} \xi_{s,t}(x) = -\frac{\partial_{yy} Y_{s,t}}{(\partial_y Y_{s,t})^3}(\xi_{s,t}(x)),$$

we easily deduce that (14) rewrites

$$\mathrm{d}\xi_{s,t}(x) = \Big[- \frac{\big[b(t,x) - \partial_x \gamma(t,x) \cdot \gamma(t,x)\big]}{(\partial_y Y_{s,t})(\xi_{s,t}(x))} - \frac{1}{2} \frac{\partial_{yy} Y_{s,t}}{(\partial_y Y_{s,t})^3}(\xi_{s,t}(x)) |\gamma(t,x)|^2 \Big] \mathrm{d}t \\ - \frac{\gamma(t,x)}{(\partial_y Y_{s,t})(\xi_{s,t}(x))} \cdot \mathrm{d}B_t.$$

This is a SDE but with stochastic coefficients. Designing a simulation scheme is quite delicate. Indeed,

- the local Lipschitz constants of the stochastic coefficients are unbounded and delicate to control; it rules out the use of standard Euler schemes;
- these coefficients depend on $\partial_y Y_{s,t}$ and $\partial_{yy} Y_{s,t}$ and their evaluation requires to compute the solution $y \mapsto Y_{s,t}(y)$ for many y. It seems to be as costly as the approach i).

iv) Inverse flow as a backward in time SDE with standard coefficients: Last, we may consider the dynamics of $\xi_{s,t}(x)$ in the variable s: doing so, we aim at computing the inverse of Y backward in time instead of forward in time. This approach relies on the following key result.

Lemma (H. Kunita, Theorem 4.2.10 p.131).

Under Assumptions of Lemma 26, the inverse flow $\xi_{s,t}(x)$ is also a semimartingale with respect to s (for any given t, x) and satisfies the following *backward SDE* (using the Backward Brownian motion $\overleftarrow{B}$)

$$\mathrm{d}\xi_{s,t}(x) = -\left[b(s,\xi_{s,t}(x)) - \partial_x\gamma(s,\xi_{s,t}(x))\cdot\gamma(s,\xi_{s,t}(x))\right]\mathrm{d}s - \gamma(s,\xi_{s,t}(x))\cdot\mathrm{d}\overleftarrow{B}_s,$$

for $s \leq t$ and with $\xi_{t,t}(x) = x$.

From this, the approximation of $\xi_{s,t}$ is made possible simply using a standard Euler scheme with time step T/N,

- Set $\xi^N_{t,t}(x) = x$ and let t_N be defined by $t_N = k_N \frac{T}{N}$ with $k_N \in \mathbb{N}$ and $t_N < t \leq t_N + \frac{T}{N}$;
- For $s \in (t_N, t]$, set

$$\xi^N_{s,t}(x) = x - \left[b(t,x) - \partial_x \gamma(t,x) \cdot \gamma(t,x)\right](t-s) - \gamma(t,x) \cdot (B_t - B_s);$$

- For $k \leq k_N$ and $s \in ((k-1)\frac{T}{N}, k\frac{T}{N}]$, set

$$\xi^N_{s,t}(x) = \xi^N_{k\frac{T}{N},t}(x) - \left[b(k\frac{T}{N}, \xi^N_{k\frac{T}{N},t}(x)) - \partial_x \gamma(k\frac{T}{N}, \xi^N_{k\frac{T}{N},t}(x)) \cdot \gamma(k\frac{T}{N}, \xi^N_{k\frac{T}{N},t}(x))\right](k\frac{T}{N} - s)$$

$$- \gamma(k\frac{T}{N}, \xi^N_{k\frac{T}{N},t}(x)) \cdot (B_{k\frac{T}{N}} - B_s).$$

Theorem 5.

Assume that

- the coefficients (μ, σ) of the SDE X satisfy Assumptions (**HP1**), (**HP2**), (**HP3**) and (**HP4**) (which α-parameter is denoted by α^X),
- the coefficients $(b - \partial_x \gamma \cdot \gamma, \gamma)$ of $\xi_{.,t}$ satisfy Assumptions (**HP1**) and (**HP3**) (which α-parameter is denoted by α^Y).

Then, for any concave function u with Lipschitz marginal utility u_x, the compound Euler scheme $X_{.}^N(u_x(\xi_{.}^N))$ converges to $U_x(.,.)$ (solution to the SPDE of the form (1)) in any $\mathbf{L}_p$-norm, at the order $\beta := \min(\alpha^X, \alpha^Y, \frac{1}{2})$ w.r.t. N: For any $p > 0$ and any $t \in [0, T]$,

$$\left\| X_{0,t}^N(u_x(\xi_{0,t}^N(x))) - U_x(t,x) \right\|_{\mathbf{L}_p} = O(N^{-\beta}).$$

APPLICATIONS TO STOCHASTIC PROCESSES

♣ Application to unbiased simulation scheme: Multi-Level Monte Carlo (MLMC) methods.

♣ Application to stochastoc process:

(i) THE CASE OF SEMIMARTINGALES AT RANDOM TIMES
- Example 1: Martingale at random times.
- Example 2: Local times at random time and random level.

(ii) THE CASE OF NON-SEMIMARTINGALES.
- Example 1: Fractional Brownian motion at random times.
- Example 2: Diffusion process in Brownian time.

♣ Backward resampling of Euler schemes.

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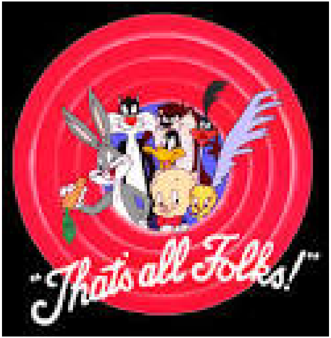

Figure: Thank you for your attention