Optimal portfolio liquidation in target zone models and catalytic superprocesses

Eyal Neuman

Imperial College London

Joint work with Alexander Schied

Outline

The Control Process

1. $S = \{S(t)\}_{t \geq 0}$ is a diffusion process with $S(0) = z$, reflected at some barrier $c \in \mathbb{R}$.

2. $L = \{L_t\}_{t \geq 0}$ is the local time of S at c.

The Control Process

1. $S = \{S(t)\}_{t \geq 0}$ is a diffusion process with $S(0) = z$, reflected at some barrier $c \in \mathbb{R}$.

2. $L = \{L_t\}_{t \geq 0}$ is the local time of S at some $c \in \mathbb{R}$.

3. Let $\mathscr{X}$ denote the class of all progressively measurable control processes ξ for which $\int_0^T |\xi_t| \, dL_t < \infty$ P_z-a.s. for all $T > 0$ and $z \in \mathbb{R}$.

4. For $\xi \in \mathscr{X}$ and $x_0 \in \mathbb{R}$ we define

$$X_t^\xi := x_0 + \int_0^t \xi_s \, dL_s, \ t \geq 0.$$

The Control Problem

We consider the minimization of the cost functional for some $p \geq 2$,

$$E_z\Big[\int_0^T |\xi_t|^p \, L(dt) + \int_0^T \phi(S_t)|X_t^\xi|^p \, dt + \varrho(S_T)|X_T^\xi|^p \Big]$$

1. For $\xi \in \mathscr{X}$ and

$$X_t^\xi := x_0 + \int_0^t \xi_s \, dL_s, \; t \geq 0.$$

The Control Problem

We consider the minimization of the cost functional for some $p \geq 2$,

$$E_z\Big[\int_0^T |\xi_t|^p \, L(dt) + \int_0^T \phi(S_t)|X_t^\xi|^p \, dt + \varrho(S_T)|X_T^\xi|^p \Big]$$

1. ϕ is a bounded measurable function.
2. $\varrho \geq 0$ is a bounded continuous penalty function.

Financial Motivation

Reflected Processes and Target Zone Models

1. Reflecting diffusion processes often be used in models for currency exchange rates in a target zone.

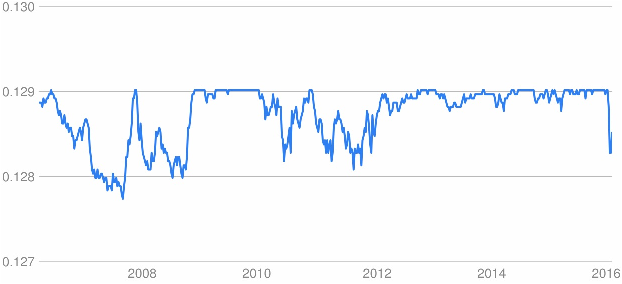
Financial Motivation: Reflected Processes and Target Zone Models

1. Reflecting diffusion processes often be used in models for currency exchange rates in a target zone.
2. **A target zone refers to a regime in which the exchange rate of a currency is kept within a certain range of values, either through an international agreement or through central bank intervention.**

Financial Motivation: Reflected Processes and Target Zone Models

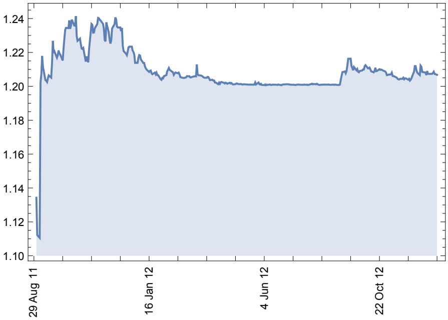
2. A target zone refers to a regime in which the exchange rate of a currency is kept within a certain range of values, either through an international agreement or through central bank intervention.
3. **See for example: Krugman (1991), Svensson (1991), Bertolla (1991), Bertolla and Caballero (1992), De Jong (1994), and Ball and Roma (1998).**

Target zone models: HKD/USD



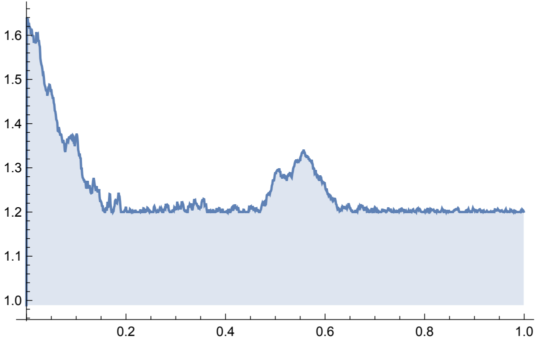
Plot of the HKD/USD exchange rate from 2007 until 2016 (currencyconverter.io).

Target zone models: EUR/CHF



Plot of the EUR/CHF exchange rate from September 1, 2011 through December 31, 2012.

Target zone models: reflected geometric Brownian motion



Plot of reflected geometric Brownian motion reflected at c=1.2.

Financial Motivation: The Control Process

$$X_t^{\xi} = x_0 + \int_0^t \xi_s \, dL_s, \ 0 \le t \le T.$$

1. The control ξ will be interpreted as a trading strategy that executes orders at infinitesimal rate $\xi_t \, dL_t$ at those times t at which $S_t = c$.

Financial Motivation: The Control Process

$$X_t^{\xi} = x_0 + \int_0^t \xi_s \, dL_s, \; 0 \leq t \leq T.$$

1. The control ξ will be interpreted as a trading strategy that executes orders at infinitesimal rate $\xi_t \, dL_t$ at those times t at which $S_t = c$.
2. **For instance, for an investor wishing to sell Swiss francs during the period of a lower bound on the EUR/CHF exchange rate.**

Financial Motivation: The Control Process

1. The control ξ will be interpreted as a trading strategy that executes orders at infinitesimal rate $\xi_t \, dL_t$ at those times t at which $S_t = c$.
2. For instance, for an investor wishing to sell Swiss francs during the period of a lower bound on the EUR/CHF exchange rate.
3. **The resulting process $X_t^\xi = x_0 + \int_0^t \xi_s \, dL_s$ describes the inventory of the investor at time t.**

Financial Motivation: The Control Problem

Recall that we wish to minimize of the cost functional for some $p \geq 2$,

$$E_z\Big[\int_0^T |\xi_t|^p \, L(dt) + \int_0^T \phi(S_t)|X_t^\xi|^p \, dt + \varrho(S_T)|X_T^\xi|^p \Big]$$

1. The expectation of $\int_0^T \phi(S_t)|X_t^\xi|^p \, dt$ can be regarded as a measure for the risk associated with holding the position X_t^ξ at time t.

Financial Motivation: The Control Problem

Recall that we wish to minimize of the cost functional for some $p \geq 2$,

$$E_z\Big[\int_0^T |\xi_t|^p \, L(dt) + \int_0^T \phi(S_t)|X_t^\xi|^p \, dt + \varrho(S_T)|X_T^\xi|^p \Big]$$

1. The expectation of $\int_0^T \phi(S_t)|X_t^\xi|^p \, dt$ can be regarded as a measure for the risk associated with holding the position X_t^ξ at time t.
2. See [Almgren 2012, Forsyth et al 2012, Tse et al 2013, Schied 2013].

Financial Motivation: The Control Problem

Recall that we wish to minimize of the cost functional for some $p \geq 2$,

$$E_z\Big[\int_0^T |\xi_t|^p \, L(dt) + \int_0^T \phi(S_t)|X_t^\xi|^p \, dt + \varrho(S_T)|X_T^\xi|^p\Big]$$

3. Similarly, the expectation of the term $\varrho(S_T)|X_T^\xi|^p$ can be viewed as a penalty for still keeping the position X_T^ξ at the end of the trading horizon.

Financial Motivation: The Control Problem

Recall that we wish to minimize of the cost functional for some $p \geq 2$,

$$E_z\Big[\int_0^T |\xi_t|^p \, L(dt) + \int_0^T \phi(S_t)|X_t^\xi|^p \, dt + \varrho(S_T)|X_T^\xi|^p\Big]$$

4. The term $\int_0^T |\xi_t|^p \, L(dt)$ can be interpreted as a cost term that arises from the temporary price impact generated by executing the strategy X^ξ.

Temporary Impact Costs Caused by $\{X_t^\xi\}_{t\geq 0}$

We focus in the case where $\{S_t\}_{t\geq 0}$ is a Brownian motion with drift.

For $n \in \mathbb{N}$ fixed, we define the following stopping times

$$\tau_0^{(n)} := \inf\{t \geq 0 \,|\, S_t \in c + 2^{-n}\mathbb{Z}\},$$

$$\tau_k^{(n)} := \inf\{t > \tau_{k-1}^{(n)} \,|\, |S_t - S_{\tau_{k-1}^{(n)}}| = 2^{-n}\}.$$

Then we introduce the discretized price process

$$S_k^{(n)} := S_{\tau_k^{(n)}}, \qquad k = 0, 1, \ldots$$

Temporary Impact Costs Caused by $\{X_t^\xi\}_{t\geq 0}$

Then we introduce the discretized price process

$$S_k^{(n)} := S_{\tau_k^{(n)}}, \qquad k = 0, 1, \ldots$$

The local time of $S^{(n)}$ in c is usually defined as

$$\ell_k^{(n)} := 2^{-n} \sum_{i=0}^{k} \mathbb{1}_{\{S_i^{(n)} = c\}}$$

Then we introduce the discretized price process

$$S_k^{(n)} := S_{\tau_k^{(n)}}, \qquad k = 0, 1, \ldots$$

The local time of $S^{(n)}$ in c is usually defined as

$$\ell_k^{(n)} := 2^{-n} \sum_{i=0}^{k} \mathbb{1}_{\{S_i^{(n)} = c\}}.$$

A generalisation of a result in [Le Gall 1994] gives us

$$S^{(n)}_{\lfloor 2^{2n} t \rfloor} \to S_t \qquad \text{and} \qquad \ell^{(n)}_{\lfloor 2^{2n} t \rfloor} \to L_t, \text{ uniformly in } t, P-\text{a.s.}$$

Temporary Impact Costs Caused by $\{X_t^\xi\}_{t\geq 0}$

Define

$$\xi_k^{(n)} := \xi_{\tau_k^{(n)}} \qquad \text{and} \qquad X_N^{\xi,(n)} := x_0 + \sum_{k=0}^{N} \xi_k^{(n)} (\ell_k^{(n)} - \ell_{k-1}^{(n)})$$

1. $\xi^{(n)}$ is the speed, relative to the local time $\ell^{(n)}$, at which shares are sold or purchased.
2. $X_N^{\xi,(n)}$ is the inventory of the investor at the N^{th} time step of the discrete-time approximation.

Temporary Impact Costs Caused by $\{X_t^{\xi}\}_{t \geq 0}$

1. Each executed order generates a temporary price impact, which is given by a function g of the trading speed (as in the framework of the Almgren-Chriss model)

Temporary Impact Costs Caused by $\{X_t^{\xi}\}_{t \geq 0}$

1. Each executed order generates a temporary price impact, which is given by a function g of the trading speed (as in the framework of the Almgren-Chriss model)
2. **Here we take $g(x) = \text{sign}(x)|x|^{p-1}$ for some $p > 1$, which is consistent with [Almgren 03, Almgren, Hauptman, Li 05, Gatheral 10].**

Temporary Impact Costs Caused by $\{X_t^{\xi}\}_{t \geq 0}$

1. Each executed order generates a temporary price impact, which is given by a function g of the trading speed (as in the framework of the Almgren-Chriss model)

2. Here we take $g(x) = \text{sign}(x)|x|^{p-1}$ for some $p > 1$, which is consistent with [Almgren 03, Almgren, Hauptman, Li 05, Gatheral 10].

3. **The speed of the k^{th} order is $\xi_k^{(n)}$ and the number of shares executed by that order is $\xi_k^{(n)}(\ell_k^{(n)} - \ell_{k-1}^{(n)})$.**

Temporary Impact Costs Caused by $\{X_t^\xi\}_{t\geq 0}$

2. Here we take $g(x) = \mathrm{sign}(x)|x|^{p-1}$ for some $p > 1$.
3. Since the speed of the k^{th} order is $\xi_k^{(n)}$ and the number of shares executed by that order is $\xi_k^{(n)}(\ell_k^{(n)} - \ell_{k-1}^{(n)})$.
4. **It follows that the total transaction costs incurred by the first N orders are equal to**

$$\sum_{k=0}^{N} |\xi_k^{(n)}|^p(\ell_k^{(n)} - \ell_{k-1}^{(n)}).$$

Temporary Impact Costs Caused by $\{X_t^\xi\}_{t\geq 0}$

The following result now provides the financial interpretation of the cost minimization problem. We assume here that ξ_t has a P_z-a.s continuous version.

Proposition

Under the above assumptions, we have that P_z-a.s. for each $t > 0$,

$$X^{\xi,(n)}_{\lfloor 2^{2n}t \rfloor} \longrightarrow X^\xi_t \quad \text{and} \quad \sum_{k=0}^{\lfloor 2^{2n}t \rfloor} |\xi_k^{(n)}|^p (\ell_k^{(n)} - \ell_{k-1}^{(n)}) \longrightarrow \int_0^t |\xi_s|^p \, L(ds).$$

Financial Motivation: The Control Problem

Recall that we wish to minimize of the cost functional for some $p \geq 2$,

$$C([0,T]) = E_z\left[\int_0^T |\xi_t|^p \, L(dt) + \int_0^T \phi(S_t)|X_t^\xi|^p \, dt + \varrho(S_T)|X_T^\xi|^p\right]$$

- The term $\int_0^T |\xi_t|^p \, L(dt)$ can be interpreted as a cost term that arises from the temporary price impact generated by executing the strategy X^ξ.

What is a superprocesses ?

1. Define $\{S^i_t\}_{i=1}^{N(t)}$ - a collection of critical branching diffusion particles that live in $\mathbb{R}$.

2. $N(t)$ is the number of the particles in the system at time t.

3. We assume that between branching events the particles follow independent diffusion paths which are independent.

What is a superprocesses ?

1. Define $\{S_t^i\}_{i=1}^{N(t)}$ - a collection of critical branching diffusion particles that live in $\mathbb{R}$.
2. $N(t)$ is the number of the particles in the system at time t.
3. We assume that between branching events the particles follow independent diffusion paths which are independent.
4. Critical branching means that each particle splits into two or dies with equal probability (independently of other particles).

What is a superprocesses ?

3. We assume that between branching events the particles follow independent diffusion paths which are independent.

4. Critical branching means that each particle splits into two or dies with equal probability (independently of other particles).

5. **We assume that the times between branching are independently distributed exponential random variables with mean $1/m$.**

6. **In what follows m is "large" (fast branching), $N(0) \sim m$.**

What is a superprocesses ?

1. We define the following measure valued process

$$Y_t^{(m)}(A) = \frac{1}{m} \sum_{i=1}^{N(t)} \delta_{S_t^{(i)}}(A), \quad A \subset \mathbb{R}.$$

   Here δ_x is the delta measure centred at x.

2. Suppose that $\{Y_0^m\}_{m\geq 1}$ converges weakly to μ, as $m \to \infty$.
3. In the appropriate topology, $\{Y_t^m\}_{t\geq 0}$ converges weakly to a limiting process $\{Y_t\}_{t\geq 0}$, which is called superporcess.

What is a Catalytic-Superprocesses ?

(with a single point catalyst at c)

We assume that the probability that a particle survives between $[r, t]$ and dies between $[t, t + dt]$ is given by

$$e^{-L(r,t)} dL(t),$$

where $\{L(t)\}_{t \geq 0}$ is the local time that the particle spends at the point c between $[0, t]$.

Unique Solution to the Control Problem

1. Consider the catalytic superprocess Y_t with a single point catalyst at c.

Unique Solution to the Control Problem

1. Consider the catalytic superprocess Y_t with a single point catalyst at c.

2. Let

$$u(t,z) := -\log \mathbb{E}_{\delta_z}\Big[\exp\Big(-\int_0^t \langle \phi, Y_s\rangle\, ds - \langle \varrho, Y_t\rangle\Big)\Big].$$

Unique Solution to the Control Problem

1. Consider the catalytic superprocess Y_t with a single point catalyst at c.

2. Let

$$u(t,z) := -\log \mathbb{E}_{\delta_z}\Big[\exp\Big(-\int_0^t \langle \phi, Y_s\rangle\, ds - \langle \varrho, Y_t\rangle\Big)\Big].$$

3. Recall that $\mathscr{X}$ denote the class of all progressively measurable control processes ξ for which $\int_0^T |\xi_t|\, dL_t < \infty$ P_z-a.s. for all $T > 0$ and $z \in \mathbb{R}$.

Theorem (N. and Schied 2016)

Let $\beta := 1/(p-1)$ and

$$\xi_t^* := -x_0 \exp\Big(-\int_0^t u(T-s, S_s)^\beta \, dL_s\Big) u(T-t, S_t)^\beta$$

so that

$$X_t^{\xi^*} = x_0 \exp\Big(-\int_0^t u(T-s, S_s)^\beta \, dL_s\Big).$$

Then ξ^* is the **unique strategy** in $\mathscr{X}$ minimizing the cost functional.

Moreover, the minimal cost is given by

$$C([0,T]) = |x_0|^p u(T, z).$$

Current Research

1. **The central bank point of view:** for any given trader strategy $\xi \in \mathscr{X}$, the actual price process is

$$\tilde{S}_t^\xi = S_t + \gamma(X_t^\xi - x_0).$$

What is the optimal strategy of the central bank which keeps $\{\tilde{S}_t^\xi\}_{\geq 0}$ above level c ?

Current Research

1. The central bank point of view: for any given trader strategy $\xi \in \mathscr{X}$, the actual price process is

$$\tilde{S}_t^{\xi} = S_t + \gamma(X_t^{\xi} - x_0).$$

   What is the optimal strategy of the central bank which keeps $\{\tilde{S}_t^{\xi}\}_{\geq 0}$ above level c ?

2. **Formulation of the trader-central bank system as a stochastic game.**

Current Research

1. The central bank point of view: for any given trader strategy $\xi \in \mathscr{X}$, the actual price process is

$$\tilde{S}_t^{\xi} = S_t + \gamma(X_t^{\xi} - x_0).$$

   What is the optimal strategy of the central bank which keeps $\{\tilde{S}_t^{\xi}\}_{\geq 0}$ above level c ?

2. Formulation of the trader-central bank system as a stochastic game.

3. **Is there an equilibrium between the central bank and the trader's optimal strategies ?**


THANK YOU
for your
ATTENTION!

Connection between Control and Superprocesses

1. Super-Brownian motion $\{Y_t\}_{t\geq 0}$ satisfies

$$E_\mu[e^{-\langle Y_t,\phi\rangle}] = e^{-\langle v(r,\cdot),\mu\rangle},$$

for every test function ϕ.

Connection between Control and Superprocesses

1. Super-Brownian motion $\{Y_t\}_{t\geq 0}$ satisfies

$$E_\mu[e^{-\langle Y_t,\phi\rangle}] = e^{-\langle v(r,\cdot),\mu\rangle},$$

   for every test function ϕ.

2. The log-Laplace functional v satisfies

$$\frac{\partial v}{\partial t} = \frac{1}{2}\Delta v - v^2, \quad v|_{t=0+} = \phi.$$

Connection Between Control and Superprocesses

In [Schied, 2013] the following value function was introduced

$$V(t, z, x_0) := \inf_{x()} E_{t,z}\Big[\int_t^T |\dot{x}(u)|^2 du + \int_t^T |x(u)|^2 a(W_u) du\Big].$$

1. Here W is a standard Brownian motion and a is some positive measurable function.

Connection Between Control and Superprocesses

In [Schied, 2013] the following value function was introduced

$$V(t,z,x_0) := \inf_{x()} E_{t,z}\Big[\int_t^T |\dot{x}(u)|^2 du + \int_t^T |x(u)|^2 a(W_u) du\Big].$$

1. Here W is a standard Brownian motion and a is some positive measurable function.
2. **The infimum is taken over the class of all absolutely continues adapted strategies $x()$ such that $x(t) = x_0$ and $x(T) = 0$.**

Connection Between Control and Superprocesses

In [Schied, 2013] the following value function was introduced

$$V(t,z,x_0) := \inf_{x()} E_{t,z}\Big[\int_t^T |\dot{x}(u)|^2 du + \int_t^T |x(u)|^2 a(W_u) du\Big].$$

The associated HJB equation is

$$V_t(t,z,x_0) + \inf_{\zeta}\left\{|\zeta|^2 + V_{x_0}(t,z,x_0)\zeta\right\} + a(z)|x_0|^2 + \frac{1}{2}\Delta V(t,z,x_0) = 0.$$

Connection Between Control and Superprocesses

In [Schied, 2013] the following value function was introduced

$$V(t,z,x_0) := \inf_{x()} E_{t,z}\Big[\int_t^T |\dot{x}(u)|^2 du + \int_t^T |x(u)|^2 a(W_u) du\Big].$$

The associated HJB equation is

$$V_t(t,z,x_0) + \inf_{\zeta}\{|\zeta|^2 + V_{x_0}(t,z,x_0)\zeta\} + a(z)|x_0|^2 + \frac{1}{2}\Delta V(t,z,x_0) = 0,$$

with $V(T,z,x_0) = 0$ if $x_0 = 0$ and $V(T,z,x_0) = \infty$ otherwise.

Connection Between Control and Superprocesses

The associated HJB equation is

$$V_t(t,z,x_0)+\inf_{\zeta}\{\zeta^2+V_{x_0}(t,z,x_0)\zeta\}+a(z)|x_0|^2+\frac{1}{2}\Delta V(t,z,x_0)=0,$$

with $V(T,z,x_0)=0$ if $x_0=0$ and $V(T,z,x_0)=\infty$ otherwise.

For $x_0>0$, assume that $V(t,z,x_0)=x_0^2 v(t,z)$ for some function v.

Connection between control problems and superprocesses

If we minimize over ζ we get that v formally stratifies:

$$\begin{cases} \frac{\partial v}{\partial t} = -\frac{1}{2}\Delta v + v^2 - a, \\ \\ v(T,z) = +\infty. \end{cases}$$

The Log-Laplace functional of SBM with branching rate 1 satisfies

$$\frac{\partial v}{\partial t} = \frac{1}{2}\Delta v - v^2, \quad v|_{t=0+} = \phi.$$

Questions?