Statistical estimation of the Oscillating Brownian Motion and application to volatility modeling

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1. Oscillating Brownian Motion, local time

2. An estimator based on quadratic variation

3. Application to volatility modeling
Consider the process in $\mathbb{R}$ solution of

$$Y_t = Y_0 + \int_0^t \sigma(Y_t) dW_t$$

with

$$\sigma(x) = \begin{cases} 
\sigma_+ & \text{for } x \geq 0 \\
\sigma_- & \text{for } x < 0
\end{cases}$$

The process is defined using the recipe of Ito-McKean to construct a process with given speed measure and scale function. It behaves like a Brownian motion which changes variance parameter each time it crosses 0. In this talk we also suppose $Y_0 = 0$ a.s., and fix the final time horizon $T = 1$.

The aim of the present work is to propose and analyze some estimators for the parameters of such process.
Tanaka formula and local time

For any continuous semimartingale $M$

\[ |M_t| - |M_0| = \int_0^t \text{sgn}(M_s) dM_s + L_t^M(0) \]

With $x^+ = x \vee 0$; $x^- = (-x) \vee 0$, we also have

\[ M_t^+ - M_0^+ = \int_0^t 1(M_s \geq 0) dM_s + \frac{1}{2} L_t^M(0) \]

\[ M_t^- - M_0^- = -\int_0^t 1(M_s < 0) dM_s + \frac{1}{2} L_t^M(0) \]

where

\[ L_t^M(0) := \lim_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon} \int_0^t 1(|M_s| \leq \varepsilon) ds \]

in the local time of $M$ at 0.
We apply the formula for the positive part to the OBM $Y$:

$$Y_t^+ = \int_0^t 1(Y_s \geq 0) \, dY_s + \frac{1}{2} \mathcal{L}^Y_t(0)$$

$$= \sigma_+ \int_0^t 1(Y_s \geq 0) \, dW_s + \frac{1}{2} \mathcal{L}^Y_t(0)$$

We hope to recover an estimator for $\sigma_+$ from the martingale part.
Approximation of quadratic variation

For fixed \( n \in \mathbb{N} \), we consider the time grid \( 0, \frac{1}{n}, \frac{2}{n}, \ldots 1 \). For any processes \( M, \tilde{M} \) we set

\[
[M, \tilde{M}]_1^n = \sum_{k=1}^{n} (M_k/n - M_{(k-1)}/n) (\tilde{M}_k/n - \tilde{M}_{(k-1)}/n).
\]

This is an estimator of the quadratic covariation of \( M, \tilde{M} \). We also write

\[
[M]_1^n = \sum_{k=1}^{n} (M_k/n - M_{(k-1)}/n)^2,
\]

and this is a classic estimator of the quadratic variation.
We set

\[ \xi_t = \int_0^t 1(Y_s \geq 0)\sigma(Y_s)\,dW_s = \sigma_+ \int_0^t 1(Y_s \geq 0)\,dW_s \]

This is a martingale with quadratic variation

\[ \langle \xi \rangle_t = \int_0^t \sigma(Y_s)^2 1(Y_s \geq 0)\,ds = \sigma_+^2 \int_0^t 1(Y_s \geq 0)\,ds \]

From classic results on martingales (Discretization of processes, Jacod, Protter, 2012), we have the following convergences for \( n \to \infty \):

\( (LLN) \quad [\xi]^n \xrightarrow{p} \langle \xi \rangle_1 = \sigma_+^2 \int_0^1 1(Y_s \geq 0)\,ds \)

\( (CLT) \quad \sqrt{n} ([\xi]^n - \langle \xi \rangle_1) \xrightarrow{s.l.} \sqrt{2}\sigma_+ \int_0^1 1(Y_s \geq 0)\,dB_s \)

where \( B \) is an independent BM.
Estimation on $Y^+$

With our definition of $\xi$,

$$Y_t^+ = \xi_t + \frac{1}{2} L_t^Y$$

We do not observe $\xi$ but $Y^+$. We have

$$[Y^+]_t = [\xi]_t - \frac{[L^Y]_t}{4} + [Y^+, L^Y]_t.$$  

$L_t^Y$ is increasing and does not contribute to the limit

$$[Y^+]_1 \to \sigma_+^2 \int_0^1 \mathbf{1}(Y_s \geq 0) ds = \sigma_+^2 Q_1^+$$

where we set $Q_1^+ = \text{Leb}\{s \in [0,1] : Y_s \geq 0\}$. For $0 < u < 1$

$$P(Q_1^+ \in du) = \frac{1}{\pi} \frac{1}{\sqrt{u(1-u)}} \frac{\sigma_+ / \sigma_-}{1 - (1 - (\sigma_+ / \sigma_-)^2)u} du.$$
Riemann sums

\[ \bar{Q}_1^n(Y, +) = \sum_{k=1}^n \frac{1(Y_{k/n} \geq 0)}{n} \]

converge a.s. to the Lebesgue integral

\[ \bar{Q}_1^n(Y, +) \xrightarrow{a.s.} \int_0^1 1(Y_s \geq 0)ds = Q_1^+ . \]

We define now \( \hat{\sigma}^+_n \), the estimator for \( \sigma_+ \), as

\[ (\hat{\sigma}^+_n)^2 = \frac{[Y]^+_n}{\bar{Q}_1^n(Y, +)} \]

We can define analogously \( \hat{\sigma}^-_n \), the estimator for \( \sigma_- \). We have

\[ \hat{\sigma}^n = (\hat{\sigma}^+_n, \hat{\sigma}^-_n) \xrightarrow{p} (\sigma_+, \sigma_-) \]
Our estimator for $\sigma_+$ is

$$(\hat{\sigma}_n^+)^2 = \frac{[Y^+]_1^n}{\bar{Q}_1^n(Y, +)}$$

Problem: in CLT, convergence in law! Cannot divide by a random sample size.

Stable convergence in law
Stable convergence (Rényi)

\[ n \in \mathbb{N}, \ Z_n \text{ r.v defined on the same probability space } (\Omega, \mathcal{F}, \mathbb{P}) \]

\[ Z_n \text{ converges stably in law to } Z \text{ if:} \]

\[ \mathbb{E} Yf(Z_n) \to \tilde{\mathbb{E}} Yf(Z) \]

\( (Z \text{ is a random variable defined on an extension, } (\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})) \)

for all bounded continuous functions \( f \) and all bounded random variables \( Y \) on \( (\Omega, F) \).

- Stable convergence in law implies convergence in law
- if \( Z_n \) and \( Z \), \( Y_n \) and \( Y \) are r. v. s.t.

\[ Z_n \to Z, \text{ stable in law} \quad Y_n \to Y, \text{ in probability} \]

then

\[ (Y_n, Z_n) \to (Y, Z) \quad \text{stable in law} \]
Central Limit Theorem

Estimator:

\[(\hat{\sigma}^n_+)^2 = \frac{[Y^+]_1^n}{Q^n_1(Y, +)}\]

where

\[[Y^+]_t^n = [\xi]^n_t - \frac{[L^Y]_1^n}{4} + [Y^+, L^Y]_1^n.\]

- Martingale part: stable in law convergence

\[(CLT) \quad \sqrt{n} ([\xi]^n_1 - \langle \xi \rangle_1) \xrightarrow{s_l} \sqrt{2} \int_0^t \sigma_+ 1(Y_s \geq 0) dB_s\]

- Local time?
- Occupation time?
We prove the following convergence:

$$\sqrt{n} \left( -\frac{[L_1^n]}{4} + [Y^+, L_1^n] \right) \overset{p}{\longrightarrow} -\frac{2\sqrt{2}}{3\sqrt{\pi}} \left( \frac{\sigma_+\sigma_-}{\sigma_+ + \sigma_-} \right) L_1^Y$$

adapting techniques from *Rates of convergence to the local time of a diffusion*, Jacod, 1998, and using convergence results for discretization of martingales (see for example *Limit theorem for stochastic processes*, Jacod, Shiryaev)
Summing up

\[ \sqrt{n} \left( [Y^+]_1^n - \langle \xi \rangle_1 \right) \]

\[ \overset{sl}{\rightarrow} 2 \int_0^1 \sigma_+^2 1(Y_s > 0) d\bar{B}_s - \frac{2\sqrt{2}}{3\sqrt{\pi}} \left( \frac{\sigma_+ \sigma_-}{\sigma_+ + \sigma_-} \right) L_1^Y. \]

Recall the estimator

\[ (\hat{\sigma}_+^n)^2 = \frac{[Y^+]_1^n}{Q_1^n(Y,+)} \]

where

\[ Q_1^n(Y,+)= \sum_{k=1}^n \frac{1(Y_{k/n} \geq 0)}{n} \]

is an approximation of the occupation time

\[ Q_1^+ = \text{Leb}(s \in [0,1] : Y_s \geq 0) \]
Speed of convergence for occupation time

For SDEs with smooth coefficients, the speed of convergence of the occupation time is $n^{3/4-}$ (Ngo, Ogawa), but there are no results for discontinuous coefficients. We prove that for $Y$ OBM with $Y_0 = 0$, the following convergence holds:

$$\sqrt{n} \left( \bar{Q}_1^n(Y, +) - Q_1^+ \right) \overset{p}{\to} 0$$

again with techniques involving local time and martingales.
Main theorem

The following convergence holds

\[
\sqrt{n} \left( (\hat{\sigma}_+^n)^2 - \sigma_+^2 \right) \overset{s.l.}{\rightarrow} \begin{pmatrix}
\frac{\sqrt{2}\sigma_+^2}{Q_1^+} \int_0^1 1(Y_s > 0) \, d\overline{B}_s \\
\frac{\sqrt{2}\sigma_-^2}{1-Q_1^+} \int_0^1 1(Y_s < 0) \, d\overline{B}_s \\
- \left( \frac{1}{Q_1^+} \frac{2\sqrt{2}}{1-Q_1^+} \frac{2\sqrt{2}}{3\sqrt{\pi}} \left( \frac{\sigma_- - \sigma_+}{\sigma_+ + \sigma_-} \right) \right) L_1(Y),
\end{pmatrix}
\]

where \( \overline{B} \) is a BM independent of \( Y \).

Occupation time \( \Leftrightarrow \) actual sample size
Main theorem

We can rewrite such convergence as follows:

\[
\sqrt{n} \left( \begin{pmatrix} \hat{\sigma}_+^2 - \sigma_+^2 \\ \hat{\sigma}_-^2 - \sigma_-^2 \end{pmatrix} \right) \overset{\text{d}}{\to} \begin{pmatrix} \frac{\sqrt{2}\sigma_+^2}{\sqrt{\Lambda}} \\ \frac{\sqrt{2}\sigma_-^2}{\sqrt{1-\Lambda}} \end{pmatrix} \begin{pmatrix} \mathcal{N}_1 - \frac{8}{3\sqrt{\pi}} \frac{1}{r+1} \frac{\xi\sqrt{1-\Lambda}}{\sqrt{(1-\Lambda)+\Lambda r^2}} \\ \mathcal{N}_2 - \frac{8}{3\sqrt{\pi}} \frac{1}{1/r+1} \frac{\xi\sqrt{\Lambda}}{\sqrt{\Lambda+(1-\Lambda)/r^2}} \end{pmatrix}
\]

where \( r = \sigma_+ / \sigma_- \), \( \xi, \mathcal{N}_1, \mathcal{N}_2, \Lambda \) are mutually independent, \( \xi \sim \exp(1) \), \( \mathcal{N}_1, \mathcal{N}_2 \sim \mathcal{N}(0,1) \) and \( \Lambda \) follows the modified arcsine law with density given by:

\[
p_\Lambda(\tau) = \frac{1}{\pi \tau^{1/2} (1-\tau)^{1/2}} \frac{r}{1 - (1-r^2)\tau}.
\]
An asymptotic bias is present in $\hat{\sigma}^n$. This bias has the same order ($\sim 1/\sqrt{n}$) as the ‘natural fluctuations” of the estimator. Since the local time is positive, $\hat{\sigma}_+^n$ has a probability greater than $1/2$ to be smaller than $\sigma_+$, and the same holds for $\hat{\sigma}_-^n$. 
A modified estimator

We define now a different estimator for $\sigma_+$:

$$m_+^n = \sqrt{\frac{[Y^+, Y]^n_1}{Q^n_1(Y, +)}}, \quad m_-^n = \sqrt{\frac{[Y^-, Y]^n_1}{Q^n_1(Y, -)}}$$

The following convergence holds for $n \to \infty$:

$$\sqrt{n} \left( (m_+^n)^2 - \sigma_+^2 \right) \xrightarrow{sl} \left( \frac{\sqrt{2}\sigma_+^2}{Q^n_1} \int_0^1 1(Y_s > 0) d\bar{B}_s, \frac{\sqrt{2}\sigma_-^2}{1-Q^n_1} \int_0^1 1(Y_s < 0) d\bar{B}_s \right)$$

where $\bar{B}$ is a BM independent of $Y$. 
We can rewrite such convergence as follows:

\[ \sqrt{n} \left( \frac{(m^n_+)^2 - \sigma^2_+}{(m^n_-)^2 - \sigma^2_-} \right) \xrightarrow{\text{d}} \left( \begin{array}{c} \frac{\sqrt{2}\sigma^2_+}{\sqrt{\Lambda}} \mathcal{N}_1 \\ \frac{\sqrt{2}\sigma^2_-}{\sqrt{1-\Lambda}} \mathcal{N}_2 \end{array} \right) \]

\( \mathcal{N}_1, \mathcal{N}_2, \Lambda \) are mutually independent, \( \mathcal{N}_1, \mathcal{N}_2 \sim \mathcal{N}(0, 1) \) and \( \Lambda \) follows the modified arcsine law.
Comparison between the estimators

\[ \sqrt{n}((\hat{\sigma}_+^n)^2 - \sigma_+^2) \text{ (dashed)} \text{ and } \sqrt{n}((m_+^n)^2 - \sigma_+^2) \text{ (solid)} \]
Comparison between the estimators and the theoretical limit distribution

\[ \sqrt{n}((\hat{\sigma}_+^n)^2 - \sigma_+^2) \] (dashed), \[ \sqrt{n}((m_+^n)^2 - \sigma_+^2) \] (solid) and the theoretical limit (red)
Application to volatility modeling

In the Black & Scholes Model, the detrended log-price follows

\[ dX_t = \sigma dW_t \]

with \( \sigma \) positive constant, \( W \) Brownian Motion. One possible generalization of this model is to let \( \sigma \) depend on the price variable \( X \) (local volatility model). The oscillating Brownian motion can be seen as an example of such models, with

\[ \sigma(x) = \begin{cases} \sigma_+ & \text{for } x \geq 0 \\ \sigma_- & \text{for } x < 0 \end{cases} \]

Simplest way to account of

- Leverage effect (volatility negatively correlated with the value of the stock)
- Volatility clustering
Literature on regime switching models

- Large literature on threshold models: threshold autoregressive models (TAR) and especially self exciting TAR (SETAR), H. Tong, . . .

- *Self exciting threshold interest rates models*, M. Decamps, M. Goovaerts, and W. Schoutens. Relation between the SET-Vasicek model and the OBM?

- *Filling the gaps*, A. Lipton and A. Sepp. Tiled volatility models are considered in connection with option pricing and implied volatility

Given an empirical time series $X = (X_t)_t$, we do not only estimate the parameters of the OBM, but also the threshold, using a MLE method. For a fixed threshold $r$, we consider the time series $X - r$ and estimate on it $\hat{\sigma}_+, \hat{\sigma}_-$, using our estimator. We then compute the log-likelihood

$$
\Lambda(r) = \sum_i \log p(X_i, X_{i+1}, \sigma_+, \sigma_-, r),
$$

and chose as threshold the level $\hat{r}$ maximizing $\Lambda$. 
Log-likelihood $\Lambda(r)$ for Procter & Gamble
Price and threshold for Procter & Gamble

An estimator based on quadratic variation

Application to volatility modeling

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Oscillating Brownian Motion, local time

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Log-likelihood $\Lambda(r)$ for CA Technologies Inc

![Graph](image-url)
Price and threshold for CA Technologies Inc

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Oscillating Brownian Motion, local time

An estimator based on quadratic variation

Application to volatility modeling
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Oscillating Brownian Motion, local time

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Comparison with Mota Esquivel

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<th>Stock</th>
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<th>RS</th>
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<td>P &amp; G</td>
<td>$r$</td>
<td>$\sigma_-$</td>
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Table: Estimated parameters

Comparison with Mota Esquivel
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Oscillating Brownian Motion, local time
An estimator based on quadratic variation
Application to volatility modeling

Log-likelihood $\Lambda(r)$ for S&P 500
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Oscillating Brownian Motion, local time

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Price and threshold for S&P 500

The algorithm detects the 2009 crisis!
Thanks!

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