# Statistical estimation of the Oscillating Brownian Motion and application to volatility modeling

Paolo Pigato
Joint work with Antoine Lejay

INRIA Nancy, equipe TOSCA

*Paris, 29/09/2016*

# The oscillating Brownian Motion

Consider the process in $\mathbb{R}$ solution of

$$Y_t = Y_0 + \int_0^t \sigma(Y_t) dW_t$$

with

$$\sigma(x) = \begin{cases} \sigma_+ & \text{for } x \geq 0 \\ \sigma_- & \text{for } x < 0 \end{cases}$$

The process is defined using the recipe of Ito-McKean to construct a process with given speed measure and scale function. It behaves like a Brownian motion which changes variance parameter each time it crosses 0. In this talk we also suppose $Y_0 = 0$ a.s., and fix the final time horizon $T = 1$.

The aim of the present work is to propose and analyze some estimators for the parameters of such process.

# Tanaka formula and local time



For any continuous semimartingale $M$

$$|M_t| - |M_0| = \int_0^t \operatorname{sgn}(M_s) dM_s + L_t^M(0)$$

With $x^+ = x \vee 0$; $x^- = (-x) \vee 0$, we also have

$$M_t^+ - M_0^+ = \int_0^t 1(M_s \geq 0) dM_s + \frac{1}{2} L_t^M(0)$$

$$M_t^- - M_0^- = -\int_0^t 1(M_s < 0) dM_s + \frac{1}{2} L_t^M(0)$$

where

$$L_t^M(0) := \lim_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon} \int_0^t 1(|M_s| \leq \varepsilon) ds$$

in the local time of $M$ at 0.

We apply the formula for the positive part to the OBM $Y$:

$$
\begin{aligned}
Y_t^+ &= \int_0^t 1(Y_s \geq 0)dY_s + \frac{1}{2}L_t^Y(0) \\
&= \sigma_+ \int_0^t 1(Y_s \geq 0)dW_s + \frac{1}{2}L_t^Y(0)
\end{aligned}
$$

We hope to recover an estimator for $\sigma_+$ from the martingale part.

# Approximation of quadratic variation



For fixed $n \in \mathbb{N}$, we consider the time grid $0, \frac{1}{n}, \frac{2}{n}, \ldots 1$. For any processes $M, \bar{M}$ we set

$$[M, \bar{M}]_1^n = \sum_{k=1}^{n} (M_{k/n} - M_{(k-1)/n})(\bar{M}_{k/n} - \bar{M}_{(k-1)/n}).$$

This is an estimator of the quadratic covariation of $M, \bar{M}$. We also write

$$[M]_1^n = \sum_{k=1}^{n} (M_{k/n} - M_{(k-1)/n})^2,$$

and this is a classic estimator of the quadratic variation.

We set

$$\xi_t = \int_0^t 1(Y_s \geq 0)\sigma(Y_s)dW_s = \sigma_+ \int_0^t 1(Y_s \geq 0)dW_s$$

This is a martingale with quadratic variation

$$\langle \xi \rangle_t = \int_0^t \sigma(Y_s)^2 1(Y_s \geq 0)ds = \sigma_+^2 \int_0^t 1(Y_s \geq 0)ds$$

From classic results on martingales (*Discretization of processes*, Jacod, Protter, 2012), we have the following convergences for $n \to \infty$:

$$(LLN) \quad [\xi]_1^n \xrightarrow{p} \langle \xi \rangle_1 = \sigma_+^2 \int_0^1 1(Y_s \geq 0)ds$$

$$(CLT) \quad \sqrt{n}\left([\xi]_1^n - \langle \xi \rangle_1\right) \xrightarrow{sl} \sqrt{2}\sigma_+ \int_0^1 1(Y_s \geq 0)dB_s$$

where $B$ is an independent BM.

# Estimation on $Y^+$



With our definition of $\xi$,

$$Y_t^+ = \xi_t + \frac{1}{2} L_t^Y$$

We do not observe $\xi$ but $Y^+$. We have

$$[Y^+]_t^n = [\xi]_t^n - \frac{[L^Y]_1^n}{4} + [Y^+, L^Y]_1^n.$$

$L_t^Y$ is increasing and does not contribute to the limit

$$[Y^+]_1^n \xrightarrow{p} \sigma_+^2 \int_0^1 \mathbf{1}(Y_s \geq 0) ds = \sigma_+^2 Q_1^+$$

where we set $Q_1^+ = \mathrm{Leb}\{s \in [0,1] : Y_s \geq 0\}$. For $0 < u < 1$

$$\mathbb{P}(Q_1^+ \in du) = \frac{1}{\pi} \frac{1}{\sqrt{u(1-u)}} \frac{\sigma_+/\sigma_-}{1-(1-(\sigma_+/\sigma_-)^2)u} du.$$

# Estimation of occupation time

Riemann sums

$$\bar{Q}_1^n(Y,+) = \sum_{k=1}^{n} \frac{\mathbf{1}(Y_{k/n} \geq 0)}{n}$$

converge *a.s.* to the Lebesgue integral

$$\bar{Q}_1^n(Y,+) \xrightarrow{a.s.} \int_0^1 \mathbf{1}(Y_s \geq 0)ds = Q_1^+.$$

We define now $\hat{\sigma}_+^n$, the estimator for $\sigma_+$, as

$$(\hat{\sigma}_+^n)^2 = \frac{[Y^+]_1^n}{\bar{Q}_1^n(Y,+)}$$

We can define analogously $\hat{\sigma}_-^n$, the estimator for $\sigma_-$. We have

$$\hat{\sigma}^n = (\hat{\sigma}_+^n, \hat{\sigma}_-^n) \xrightarrow{p} (\sigma_+, \sigma_-)$$

# Rate of convergence?

Our estimator for $\sigma_+$ is

$$(\hat{\sigma}_+^n)^2 = \frac{[Y^+]_1^n}{\bar{Q}_1^n(Y,+)}$$

Problem: in CLT, convergence in law! Cannot divide by a random sample size.

➥Stable convergence in law

# Stable convergence (Rényi)



$n \in \mathbb{N}$, $Z_n$ r.v defined on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$
$Z_n$ converges stably in law to $Z$ if:

$$\mathbb{E} Y f(Z_n) \to \tilde{\mathbb{E}} Y f(Z)$$

($Z$ is a random variable defined on an extension, $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\Pr})$) for all bounded continuous functions $f$ and all bounded random variables $Y$ on $(\Omega, F)$.

- Stable convergence in law implies convergence in law
- if $Z_n$ and $Z$, $Y_n$ and $Y$ are r. v. s.t.

$$Z_n \to Z, \text{ stable in law} \qquad Y_n \to Y, \text{ in probability}$$

then

$$(Y_n, Z_n) \to (Y, Z) \quad \text{stable in law}$$

# Central Limit Theorem



Estimator:

$$(\hat{\sigma}_+^n)^2 = \frac{[Y^+]_1^n}{\bar{Q}_1^n(Y,+)}$$

where

$$[Y^+]_t^n = [\xi]_t^n - \frac{[L^Y]_1^n}{4} + [Y^+, L^Y]_1^n.$$

- Martingale part: stable in law convergence

$$(CLT) \quad \sqrt{n}\left([\xi]_1^n - \langle \xi \rangle_1\right) \xrightarrow{sl} \sqrt{2} \int_0^t \sigma_+ \mathbf{1}(Y_s \geq 0) dB_s$$

- Local time?
- Occupation time?

# Local time part



We prove the following convergence:

$$\sqrt{n}\left(-\frac{[L^Y]_1^n}{4}+[Y^+,L^Y]_1^n\right)\xrightarrow{p}-\frac{2\sqrt{2}}{3\sqrt{\pi}}\left(\frac{\sigma_+\sigma_-}{\sigma_++\sigma_-}\right)L_1^Y$$

adapting techniques from *Rates of convergence to the local time of a diffusion*, Jacod, 1998, and using convergence results for discretization of martingales (see for example *Limit theorem for stochastic processes*, Jacod, Shiryaev)

Summing up

$$\sqrt{n}\big([Y^+]_1^n - \langle \xi \rangle_1\big)$$
$$\xrightarrow{sl} \sqrt{2}\int_0^1 \sigma_+^2 \mathbf{1}(Y_s > 0)\, d\bar{B}_s - \frac{2\sqrt{2}}{3\sqrt{\pi}}\left(\frac{\sigma_+\sigma_-}{\sigma_+ + \sigma_-}\right) L_1^Y.$$

Recall the estimator

$$(\hat{\sigma}_+^n)^2 = \frac{[Y^+]_1^n}{\bar{Q}_1^n(Y,+)}$$

where

$$\bar{Q}_1^n(Y,+) = \sum_{k=1}^n \frac{\mathbf{1}(Y_{k/n} \geq 0)}{n}$$

is an approximation of the occupation time

$$Q_1^+ = \text{Leb}(s \in [0,1] : Y_s \geq 0)$$

# Speed of convergence for occupation time



For SDEs with smooth coefficients, the speed of convergence of the occupation time is $n^{3/4-}$ (Ngo, Ogawa), but there are no results for discontinuous coefficients.

We prove that for $Y$ OBM with $Y_0 = 0$, the following convergence holds:

$$\sqrt{n}\left(\bar{Q}_1^n(Y,+) - Q_1^+\right) \xrightarrow{p} 0$$

again with techniques involving local time and martingales.

# Main theorem



The following convergence holds

$$\sqrt{n}\begin{pmatrix} (\hat{\sigma}_+^n)^2 - \sigma_+^2 \\ (\hat{\sigma}_-^n)^2 - \sigma_-^2 \end{pmatrix} \xrightarrow{sl} \begin{pmatrix} \frac{\sqrt{2}\sigma_+^2}{Q_1^+} \int_0^1 \mathbf{1}(Y_s > 0) d\bar{B}_s \\ \frac{\sqrt{2}\sigma_-^2}{1-Q_1^+} \int_0^1 \mathbf{1}(Y_s < 0) d\bar{B}_s \end{pmatrix} - \begin{pmatrix} \frac{1}{Q_1^+} \\ \frac{1}{1-Q_1^+} \end{pmatrix} \frac{2\sqrt{2}}{3\sqrt{\pi}} \left( \frac{\sigma_- \sigma_+}{\sigma_+ + \sigma_-} \right) L_1(Y),$$

where $\bar{B}$ is a BM independent of $Y$.

Occupation time $\Leftrightarrow$ actual sample size

# Main theorem



We can rewrite such convergence as follows:

$$\sqrt{n}\begin{pmatrix} (\hat{\sigma}_+^n)^2 - \sigma_+^2 \\ (\hat{\sigma}_-^n)^2 - \sigma_-^2 \end{pmatrix} \xrightarrow{l} \begin{pmatrix} \frac{\sqrt{2}\sigma_+^2}{\sqrt{\Lambda}}\left(\mathcal{N}_1 - \frac{8}{3\sqrt{\pi}}\frac{1}{r+1}\frac{\xi\sqrt{1-\Lambda}}{\sqrt{(1-\Lambda)+\Lambda r^2}}\right) \\ \frac{\sqrt{2}\sigma_-^2}{\sqrt{1-\Lambda}}\left(\mathcal{N}_2 - \frac{8}{3\sqrt{\pi}}\frac{1}{1/r+1}\frac{\xi\sqrt{\Lambda}}{\sqrt{\Lambda+(1-\Lambda)/r^2}}\right) \end{pmatrix}$$

where $r = \sigma_+/\sigma_-$, $\xi, \mathcal{N}_1, \mathcal{N}_2, \Lambda$ are mutually independent, $\xi \sim \exp(1)$, $\mathcal{N}_1, \mathcal{N}_2 \sim N(0,1)$ and $\Lambda$ follows the modified arcsine law with density given by:

$$p_\Lambda(\tau) = \frac{1}{\pi\tau^{1/2}(1-\tau)^{1/2}}\frac{r}{1-(1-r^2)\tau}.$$

An asymptotic bias is present in $\hat{\sigma}^n$. This bias has the same order ($\sim 1/\sqrt{n}$) as the 'natural fluctuations" of the estimator. Since the local time is positive, $\hat{\sigma}^n_+$ has a probability greater than $1/2$ to be smaller than $\sigma_+$, and the same holds for $\hat{\sigma}^n_-$.

# A modified estimator

We define now a different estimator for $\sigma_+$:

$$m_+^n = \sqrt{\frac{[Y^+, Y]_1^n}{\bar{Q}_1^n(Y,+)}}, \qquad m_-^n = \sqrt{\frac{[Y^-, Y]_1^n}{\bar{Q}_1^n(Y,-)}}$$

The following convergence holds for $n \to \infty$:

$$\sqrt{n}\begin{pmatrix} (m_+^n)^2 - \sigma_+^2 \\ (m_-^n)^2 - \sigma_-^2 \end{pmatrix} \xrightarrow{sl} \begin{pmatrix} \frac{\sqrt{2}\sigma_+^2}{Q_1^+} \int_0^1 \mathbf{1}(Y_s > 0) d\bar{B}_s \\ \frac{\sqrt{2}\sigma_-^2}{1-Q_1^+} \int_0^1 \mathbf{1}(Y_s < 0) d\bar{B}_s \end{pmatrix}$$

where $\bar{B}$ is a BM independent of $Y$.

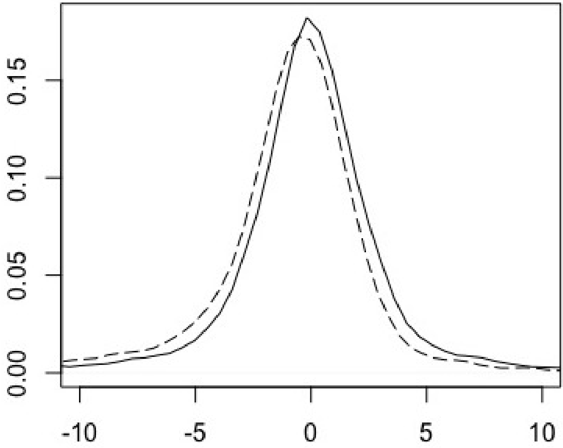
We can rewrite such convergence as follows:

$$\sqrt{n}\begin{pmatrix} (m_+^n)^2 - \sigma_+^2 \\ (m_-^n)^2 - \sigma_-^2 \end{pmatrix} \xrightarrow{l} \begin{pmatrix} \frac{\sqrt{2}\sigma_+^2}{\sqrt{\Lambda}}\mathcal{N}_1 \\ \frac{\sqrt{2}\sigma_-^2}{\sqrt{1-\Lambda}}\mathcal{N}_2 \end{pmatrix}$$

$\mathcal{N}_1, \mathcal{N}_2, \Lambda$ are mutually independent, $\mathcal{N}_1, \mathcal{N}_2 \sim N(0,1)$ and $\Lambda$ follows the modified arcsine law.

# Comparison between the estimators

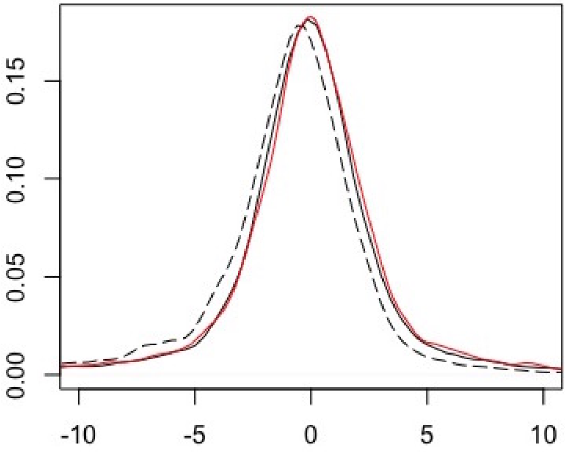
$\sqrt{n}((\hat{\sigma}_+^n)^2 - \sigma_+^2)$ (dashed) and $\sqrt{n}((m_+^n)^2 - \sigma_+^2)$ (solid)



# Comparison between the estimators and the theoretical limit distribution



$\sqrt{n}((\hat{\sigma}_+^n)^2 - \sigma_+^2)$ (dashed), $\sqrt{n}((m_+^n)^2 - \sigma_+^2)$ (solid) and the theoretical limit (red)



# Application to volatility modeling



In the Black & Scholes Model, the detrended log-price follows

$$dX_t = \sigma dW_t$$

with $\sigma$ positive constant, $W$ Brownian Motion. One possible generalization of this model is to let $\sigma$ depend on the price variable $X$ (*local volatility* model). The oscillating Brownian motion can be seen as an example of such models, with

$$\sigma(x) = \begin{cases} \sigma_+ & \text{for } x \geq 0 \\ \sigma_- & \text{for } x < 0 \end{cases}$$

Simplest way to account of

- Leverage effect (volatility negatively correlated with the value of the stock)
- Volatility clustering

# Literature on regime switching models



- Large literature on threshold models: threshold autoregressive models (TAR) and especially self exciting TAR (SETAR), H. Tong, ...
- *Self exciting threshold interest rates models*, M. Decamps, M. Goovaerts, and W. Schoutens.
  Relation between the SET-Vasicek model and the OBM?
- *Filling the gaps*, A. Lipton and A. Sepp.
  Tiled volatility models are considered in connection with option pricing and implied volatility
- *On a continuous time stock price model with regime switching, delay, and threshold*, P. P. Mota and M. L. Esquivel.

# Maximum likelihood estimation of the threshold

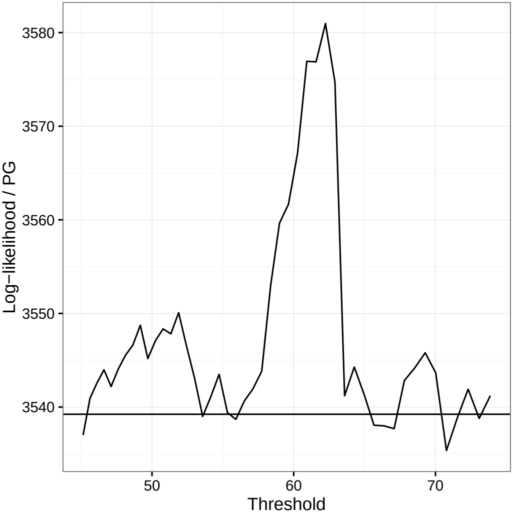


Given an empirical time series $\mathrm{X} = (\mathrm{X}_t)_t$, we do not only estimate the parameters of the OBM, but also the threshold, using a MLE method.
For a fixed threshold $r$, we consider the time series $\mathrm{X} - r$ and estimate on it $\hat{\sigma}_+, \hat{\sigma}_-$, using our estimator. We then compute the log-likelihood

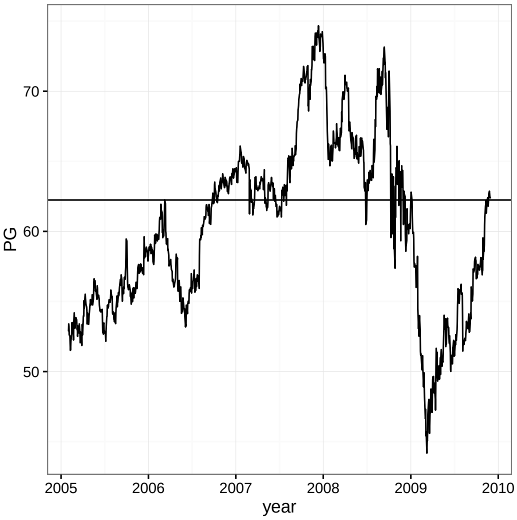
$$\Lambda(r) = \sum_i \log p(X_i, X_{i+1}, \sigma_+, \sigma_-, r),$$

and chose as threshold the level $\hat{r}$ maximizing $\Lambda$.

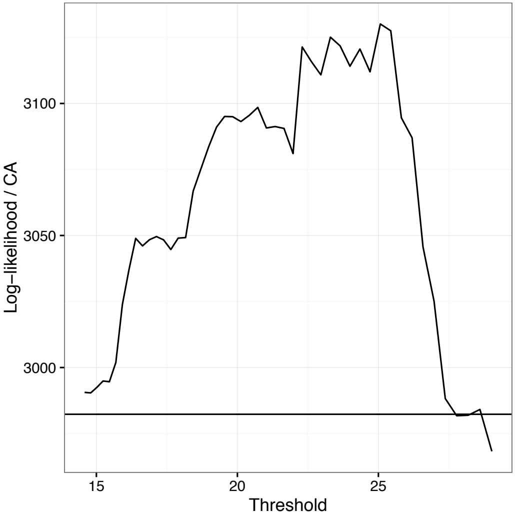
## Log-likelihood $\Lambda(r)$ for Procter & Gamble



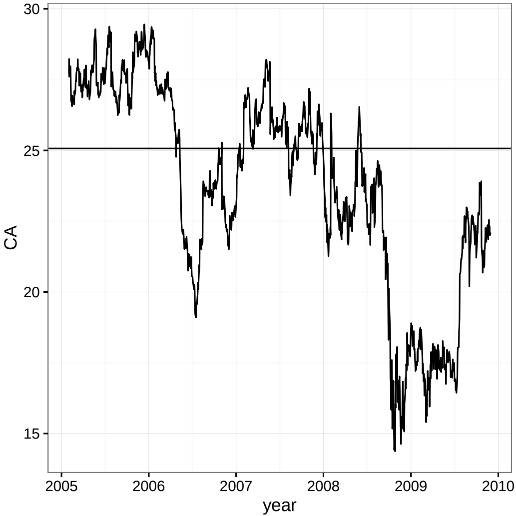
# Price and threshold for Procter & Gamble



# Log-likelihood $\Lambda(r)$ for CA Technologies Inc



## Price and threshold for CA Technologies Inc

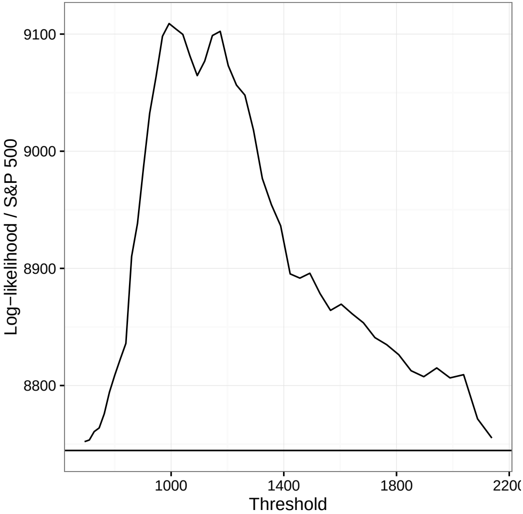


# Comparison with Mota Esquivel

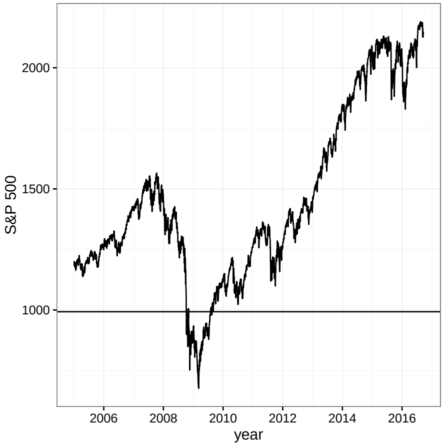
Table: Estimated parameters

| | OBM | | | RS | | |
|---|---|---|---|---|---|---|
| Stock | $r$ | $\sigma_-$ | $\sigma_+$ | $r$ | $\sigma_-$ | $\sigma_+$ |
| P & G | 62.24 | 0.014 | 0.012 | 61.9 | 0.013 | 0.013 |
| McDonalds | 52.4 | 0.013 | 0.018 | 54.6 | 0.014 | 0.016 |
| CA Inc | 25.07 | 0.025 | 0.013 | 22.16 | 0.033 | 0.015 |
| Microsoft | 21.8 | 0.034 | 0.017 | 22.8 | 0.034 | 0.016 |
| Citigroup | 40.7 | 0.075 | 0.011 | 43.1 | 0.076 | 0.011 |

# Log-likelihood $\Lambda(r)$ for S&P 500



## Price and threshold for S&P 500




The algorithm detects the 2009 crisis!

# Thanks!

- paolo.pigato@inria.fr
- Post-doc at INRIA Nancy, equipe TOSCA.