Integral Representation of Martingales in Mathematical Finance

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Problem Formulation

Given is a filtered probability space $(\Omega, \mathbf{F}, (\mathcal{F}_t)_{t\in[0,1]}, \mathbb{P})$.

Inputs:

1. $\mathbb{Q} \sim \mathbb{P}$.
2. $S = (S_t^i)$ a $\mathbb{Q}$-martingale.

Goal: conditions on S such that $\forall$ $\mathbb{Q}$-martingales M:

$$M_t = M_0 + \int_0^t H_u \,\mathrm{d}S_u, \; t \in [0,1]. \qquad \text{(MR)}$$

Theorem (Jacod 79)

(MR) $\iff$ $\mathbb{Q}$ *is ! equivalent martingale measure for* S.

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Market Completeness

In mathematical finance we typically interpret the inputs as follows:

$\mathbb{Q}$: arbitrage-free pricing measure,

$S_t = (S_t^i)$: prices of traded securities.

The martingales M correspond to replicable securities.

Theorem (Harrison & Pliska '83)
(MR) $\iff$ S-market is complete.

Verification of Market Completeness

Forward Setup

Inputs: $\mathbb{Q} \sim \mathbb{P}$, $W = (W_t^j)$ a $\mathbb{Q}$-Brownian motion, $\sigma = (\sigma_t^{ij})$.

S defined in terms of its predictable characteristics *forward* in time:

$$S_t = S_0 + \int_0^t \sigma_u \, \mathrm{d}W_u.$$

Theorem (Yor '77, Karatzas & Shreve '98)

If $\mathcal{F}_t = \mathcal{F}_t^W$, then (MR) for S $\iff \det(\sigma_t) \neq 0 \ \mathrm{d}\mathbb{P} \times \mathrm{d}t$ *a.s.*

Verification of Market Completeness

Backward Setup

Inputs: $\mathbb{Q} \sim \mathbb{P}$, $W = (W_t^j)$ $\mathbb{Q}$-Brownian motion, $\psi = (\psi^i) \in \mathcal{F}_1$.

S defined as conditional expectation *backward* in time:

$$S_t := \mathbb{E}^{\mathbb{Q}}[\psi | \mathcal{F}_t] = S_0 + \int_0^t \sigma_u \, \mathrm{d}W_u,$$

where $\sigma = (\sigma_t^{ij})$ from Brownian martingale representation.

Problem: conditions on ψ only for (MR) to hold.

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Problem: conditions on ψ only for (MR) to hold.

Literature: AR '08, HMT '12, RH '12, KP '12.

$$\mathcal{F}_t = \mathcal{F}_t^X \text{ and } \psi = g(X_1) \text{ and } \det J[g](\cdot) \neq 0 \; a.e.$$

$+$ standard assumptions $\Longrightarrow$ (MR) for S.

Verification of Market Completeness

Forward-Backward Setup

Inputs: $\mathbb{Q} \sim \mathbb{P}$, $W = (W_t^1, W_t^2)$ $\mathbb{Q}$-B.m., $\nu = \nu(\cdot)$, $h = h(\cdot)$.

$S = (S^F, S^B)$ represents prices of stock and option contract:

$$\begin{aligned} S_t^F &= S_0^F + \int_0^t \nu(W_u^2)\, \mathrm{d}W_u^1 \\ S_t^B &:= \mathbb{E}^{\mathbb{Q}}[h(S_1^F)|\mathcal{F}_t] = S_0^B + \int_0^t Z_u\, \mathrm{d}W_u. \end{aligned}$$

Verification of Market Completeness

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(MR) for (S^F, S^B)

$$\Longleftrightarrow$$

$$\det \sigma_t = \begin{vmatrix} \nu & 0 \\ Z^1 & Z^2 \end{vmatrix} \neq 0 \ a.s. \text{ BUT } \det \sigma_1 = \begin{vmatrix} \nu & 0 \\ h_s & 0 \end{vmatrix} = 0.$$

Literature: Romano & Touzi '97, Davis & Obloj '08.

Partial Radner Equilibrium

Definition

In financial economics securities are valued to lead to equilibria:

Agents: $(x^m, U^m)_{m=1}^M$.

Partial Radner Equilibrium: $((S^F, S^B), (\theta^F, \theta^B))$ such that

1. $S_1^B = \psi$,
2. given (S^F, S^B)
   (a) $U^m(x^m + \int_0^1 \theta^{F,m}\, \mathrm{d}S^F + \int_0^1 \theta^{B,m}\, \mathrm{d}S^B) \xrightarrow[\theta^{F,m},\ \theta^{B,m}]{} \max$,
   (b) $\sum_{m=1}^M \theta^{B,m} = 0$ (clearing).

Partial Radner Equilibrium

Existence

Step 1: static problem $\rightarrow$ $\mathbb{Q}$.

(a) $U^m(x^m + \int_0^1 \theta^{F,m}\, \mathrm{d}S^F + \chi^m) \xrightarrow[\theta^{F,m},\ \mathbb{E}^{\mathbb{Q}}[\chi^m]=0]{} \max$,

(b) $\sum_{m=1}^M \chi^m = 0$ (clearing).

Existence: fixed-point arguments.

$$\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}} = \text{const.} \times U_c\Big(\sum_{m=1}^{M}\big(x^m + \int_0^1 \theta^{F,m}\, \mathrm{d}S^F\big), w\Big),$$

$U(c, w)$: w-weighted sup-convolution of U^m, $w \in \operatorname{int} \Sigma^M$.

Step 2: verification of (MR) for $(S^F, S^B) \rightarrow S^B$.

$$S_t^B := \mathbb{E}^{\mathbb{Q}}[\psi | \mathcal{F}_t].$$

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Setting

Inputs: $\mathbb{Q} \sim \mathbb{P}$, $W = (W_t^1, W_t^2)$ $\mathbb{Q}$-B.m., state process X:

$$X_t = X_0 + \int_0^t b(u, X_u)\,\mathrm{d}u + \int_0^t \eta(u, X_u)\,\mathrm{d}W_u.$$

The prices of stock (S^F) and option contract (S^B) are given by

$$S_t^F = f(t, X_t),$$

and

$$S_t^B := \mathbb{E}^{\mathbb{Q}}[h(X_1)|\mathcal{F}_t].$$

Problem: conditions on b, η, f and h such that (MR) holds for $S = (S^F, S^B)$.

Conditions

$$\mathcal{B}_K(h,\varphi,t) := \int_K \frac{1}{2} A^{jk} \frac{\partial h}{\partial x^j} \frac{\partial \varphi}{\partial x^k} - (B^j - \frac{1}{2} \frac{\partial A^{jk}}{\partial x^k}) \frac{\partial h}{\partial x^j} \varphi \, \mathrm{d}x$$

Structural:

(A1) $\forall K \subset\subset \mathbb{R}^2, \ \exists \varphi \in W^1_{p,0}$ s.t. $\mathcal{B}_K[h,\varphi,1] \neq 0$.

Regularity:

(A2) $t \mapsto b(t,\cdot), \eta(t,\cdot), f(t,\cdot)$ are

- (a) analytic of $(0,1)$ to C,
- (b) continuous of $[0,1]$ to C^2.

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Example

Stochastic volatility model completed with a call option, then (A1) becomes:

$$\partial_x \nu(x) \neq 0 \text{ a.e. on } \mathbb{R}.$$

Main Result

Theorem (S. '16)

If $\mathcal{F}_t = \mathcal{F}_t^X$ and (A1) and (A2) + standard assumptions hold
$\Longrightarrow$ *(MR) for $S = (S^F, S^B)$.*

Elements of Proof

A PDE for the option price:

$$S_t^B = \mathbb{E}^{\mathbb{Q}}[h(X_1)|\mathcal{F}_t] = v(t, X_t),$$

where

$$v_t + \mathcal{L}^X(t)v = 0, \quad v(1,\cdot) = h(\cdot).$$

Evolution of security prices $S = (S^F, S^B)$:

$$\mathrm{d}S_t = (J[f,v]\eta)(t, X_t)\ \mathrm{d}W_t$$

Need to show:

$$J[f,v](t,x)$$

is nonsingular $\mathrm{d}t \times \mathrm{d}x$ a.e.

Elements of Proof

$w(t,x) := \det J[f,v](t,x)$, then

$$w_t + \mathcal{L}^X(t)w = -\mathcal{P}(t)v.$$

Evolution equations: $t \mapsto w(t,\cdot)$ is

(a) analytic on $(0,1)$,

(b) continuous on $[0,1]$.

Suppose for a contradiction $w = 0$ on open $E \subset (0,1) \times \mathbb{R}^2$.

Analyticity: $\mathcal{P}(t)v = 0$ on $(0,1)$.

Weak-formulation: $\mathcal{B}_K(v,\varphi,t) = 0$ on $(0,1)$ $\forall\varphi$.

Continuity: $\mathcal{B}_K(h,\varphi,1) = 0$ $\forall\varphi$.

$\square$

Merci Beaucoup!
Questions?